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Abstract. Probability weighting function (PWF) is the psychological probability of a decision-maker for objective probability, which reflects and predicts the risk preferences of decision-maker in behavioral decision-making. The existing approaches to PWF estimation generally include parametric methodologies to PWF construction and nonparametric elicitation of PWF. However, few of them explores the combination of parametric and nonparametric elicitation approaches to approximate PWF. To describe quantitatively risk preferences, the Newton interpolation, as a well-established mathematical approximation approach, is introduced to task-specifically match PWF under the frameworks of prospect theory and cumulative prospect theory with descriptive psychological analyses. The Newton interpolation serves as a nonparametric numerical approach to the estimation of PWF by fitting experimental preference points without imposing any specific parametric form assumptions. The elaborated nonparametric PWF model varies in accordance with the number of the experimental preference points elicitation in terms of its functional form. The introduction of Newton interpolation to PWF estimation into decision-making under risk will benefit to reflect and predict the risk preferences of decision-makers both at the aggregate and individual levels. The Newton interpolation-based nonparametric PWF model exhibits an inverse S-shaped PWF and obeys the fourfold pattern of decision-makers' risk preferences as suggested by previous empirical analyses.

Keywords: probability weighting function, risk preference, nonparametric numerical approach, Newton interpolation, preference points, decision-making under risk.

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Introduction

Since the publication of the seminal paper on prospect theory (PT) by Kahneman and Tversky (2013), PT has become one of the most prominent decision-making theories under risk and uncertainty (Blanco-Mesa et al., 2017; Chen et al., 2022; Jiang et al., 2022; Wang et al., 2023; Yu et al., 2022, 2018; Yang et al., 2022). Probability weighting function (PWF), as a main component of PT, has been widely studied and applied in a number of areas, including medical decision-making, assets portfolio, welfare lottery, organizational behavior, insurance purchase and strategic decisions, due to its capability to measure the risk attitudes of decision-makers as well as to reflect and predict their risk preferences (Huang et al., 2021; Walther & Munster, 2021). Existing studies, such as Bleichrodt and Pinto (2000), also concluded that ignoring probability weighting in modeling decision-making under risk and in utility measurement potentially produces descriptively invalid theories and distorted elicitation process. For these reasons, many researchers investigated the shaping and properties of PWF and proposed a considerable amount of creative methodologies to elicit probability weighting to promisingly reflect and predict the risk preferences of decision-makers.

The existing approaches to PWF estimation can be divided into two main categories, namely, parametric methodologies to the PWF construction (i.e., assuming specific parametric forms to accommodate the features of PWF) (Barberis, 2018; Bernheim & Sprenger, 2020; Baillon et al., 2022) and nonparametric elicitation of PWF (i.e., eliciting the preference functional of a decision-maker without imposing any prior assumptions) (Wu et al., 2021). However, to quantitatively estimate the PWF, some functional forms (e.g., one-parameter, two-parameter and multi-parameter forms) must be assumed. As a result, the shape and properties of PWF will be solely determined as per the choice of functional forms. If the true functional form is different from the assumed functional form, then it cannot flexibly reflect and predict the risk preferences of decision-maker. To circumvent this problem and attempt a new estimation with as few assumptions as possible, several researchers proposed and implemented a nonparametric approach to elicit utility function and PWF under different theoretical frameworks (i.e., rank-dependent expected utility theory (RDEU), prospect theory (PT) and cumulative prospect theory (CPT) (Chen et al., 2021; Ruggeri et al., 2020; Wang et al., 2020) without specifying their functional forms.

In this paper, a rational nonparametric PWF model-based numerical approach is presented to approximate PWF under the frameworks of PT and CPT with descriptive psychological analysis. Although this approach is nonparametric, it absorbs the advantages of parameter approach by means of utilizing parameter models to infer several computation preference points. In actual decision-making, a nonparametric PWF model is proposed to reflect the risk preferences of decision-maker, as long as some preference information of decision-maker are proved or collected. Based on the constructed PWF model, we can predict the risk preference of decision-maker so that it provides the basis for behavior decision-making. More specifically, this nonparametric numerical approach is conducted as described in the subsequent paragraph.

Firstly, a numerical approximation approach, namely, Newton interpolation, is introduced to approximate the PWF. Secondly, an elicitation method, namely, certainty equivalent meth-

od, is used to elicit the PWF under the frameworks of PT and CPT in order to collect several experimental preference points. Thirdly, in order to obtain more preference information of decision-maker from the collected experimental preference points, an appropriate parameter PWF model (prelec's PWF) is chosen to infer several computation preference points, and the parameters of prelec's PWF are determined by parameter estimate and measurement of fit. Finally, to build a nonparametric numerical model, Newton interpolation is used to approximate the obtained preference points.

This paper proposes and uses Newton interpolation approach to approximate PWF to reflect and predict the risk preferences of decision-maker both at the aggregate and individual data levels. Empirical studies apply the aggregate data from Tversky and Kahneman (1992) and the individual data from Gonzalez and Wu (1999) because of the limitations of conditions. A comparative analysis is conducted to compare the performances of prelec's two-parameter model and the proposed nonparametric numerical model. The results are consistent with the previous studies that PWF is non-linear with inverse S-shape, and has a fourfold pattern of risk preferences (i.e., overweighting for small probabilities and underweighting for big probabilities. Moreover, decision-maker is more sensitive to loss than gain and more sensitive to high than low monetary outcomes).

This paper is organized as follows: The next section presents the extant literature and its development status. Section 2 describes some preliminaries to propose a nonparametric numerical approach. Section 3 firstly introduces the Newton interpolation approach, and then a nonparametric numerical approach is proposed to build a nonparametric numerical PWF model. Section 4 elaborates on some empirical evidences based on the experimental data coming from Tversky and Kahneman (1992) and Gonzalez and Wu (1999). The last section concludes and discusses the contribution, as well as proposes the potential research directions.

1. Literature review

This paper proposed a nonparametric numerical approach to approach PWF to reflect and predict the risk preferences of decision-maker. However, different risk preferences generate different shapes of PWF. There are a lot of factors to affect the risk preferences of people, such as gender, method of obtaining information, environment and differences among countries and regions. Researchers made substantial contributions to measure the risk preferences of decision-maker in each aspect. For instance, Croson and Gneezy (2009) indicated gender differences in risk, social and competitive preferences with women being more risk-aversion than men. Von Gaudecker et al. (2011) used a large amount of experimental data to analyze the heterogeneity of risk preferences. Tanaka et al. (2010) conducted experiments in Vietnamese villages to directly measure risk and time preferences of individuals. Rieger et al. (2015) conducted a survey on risk attitudes in 53 countries to measure these cross-country differences, the main finding is that there are substantial cross-country differences in risk attitudes that depend not only on economic conditions but also on cultural factors. Roussanov and Savor (2014) explored that marital status could both reflect and affect individual preferences of chief executive officers.

To estimate risk preferences of different decision-makers, Van Ryzin and Vulcano (2015) proposed an approach for estimating preferences with defining a choice model based on a discrete probability mass function. Petrova et al. (2014) investigated the interactive influence of affective and cognitive skills on probability weighting, and studied the numerical effect on probability weighting. In this paper, we proposed a nonparametric PWF model-based numerical approach for reflecting and predicting the risk preferences of different decision-makers. The model was built by integrating parameter model and nonparametric approaches.

Tversky and Kahneman (1992) utilized a one-parameter power function to fit PWF, the obtained function encompasses weighting functions with both concave and convex parts. Based on diagonal concavity, sub-proportionality and compound invariance of PWF, Prelec (1998) found an exponential functional form. Tversky and Fox (1995) and Prelec (1998) presented a different two-parameter PWF model. They both deeded that two parameters forming the shape of PWF could well reflect its psychological preference properties, i.e., a parameter control curvature of PWF, the other parameter control its elevation. Gonzalez and Wu (1999) considered that the weighting function was not a subjective probability but rather a distortion of the given probability, they added a different psychological concept for two-parameter PWF models, i.e., discriminability and attractiveness relative to curvature and elevation. More specifically, discriminability is related to the slope of PWF, attractiveness exhibits the risk preferences of decision-maker. Brandstätter et al. (2002) presented an empirical approach to reconstruct two-parameter PWF by using the notions of elation and disappointment, that is, expected elation (anticipated elation weighted its probability of occurrence) and expected disappointment (anticipated disappointment weighted its likelihood probability of occurrence). Abdellaoui et al. (2010) offered a preference foundation for a two-parameter family of PWF and utilized two aspects of probabilistic risk attitudes' two aspects, namely, optimism and pessimism, to interpret the psychological arguments for the separate components of the curvature and elevation of PWF. Based on the dual properties of PWF, Diecidue et al. (2009) proposed a switch-power weighting function with three parameters. These three parameters control the size of concave interval relative to convex interval and the inverse S-shaped property of PWF.

The reviewed literature provided some contributions and led to several observations. Firstly, they introduced several parameter models to quantitatively analyze PWF. Secondly, the studies used psychological knowledge to interpret the parameters of PWF. Finally, empirical studies were conducted to verify the obtained PWF model's properties. Overall, the PWF of reflecting and predicting decision-makers' risk preference has an inverse S-shape, and is not linear. In particular, the evidence suggested that decision-maker always overweights low probabilities and underweights high probabilities. For further evidence, refer to Abdellaoui et al. (2010), Brandstätter et al. (2002), Gonzalez and Wu (1999), Kilka and Weber (2001), Prelec (1998), Starmer (2000), Stewart et al. (2015), Kahneman and Tversky (1984), Wu and Gonzalez (1996, 1999).

An elicitation method was early proposed by Wakker and Deneffe (1996), who designed a tradeoff method to elicit utilities in decision under risk and uncertainty. Based on the tradeoff method, Abdellaoui (2000) offered a nonparametric two-step method (equally spaced outcomes in terms of utility and equally spaced probabilities in terms of the weighting function)

to elicit utility function and PWF under RDEU and CPT. At the same time, Bleichrodt and Pinto (2000) also applied nonparametric elicitation method to elicit PWF in a new domain, namely, medical decision analysis domain. In addition, Abdellaoui et al. (2007), Booij and Van de Kuilen (2009), and Van Houtven et al. (2011) constructed a robust elicitation method to elicit PWFs under a different theoretical framework.

To summarize, several researchers have investigated the shape and properties of PWF. For instance, Prelec (1998) was devoted to deriving the observed properties of probability weighting from preference axioms, relying on the common-ratio effect as the basic building block. Diecidue et al. (2009) and Al-Nowaihi and Dhami (2010) outlined the nonlinear of PWF. Moreover, Cavagnaro et al. (2013) adopted an adaptive design optimization method to discriminate different parametric forms of the PWF. The empirical experiment shows that the PWF model of Prelec (1998) was the most common best-fitting models. Based on this reason, a nonparametric numerical approach is used to approximate PWF in this paper. Prelec's two-parameter model is used to infer computation preference pints. At the same time, elicitation methods in decision-making under risk allow researchers to infer the PWF of decision-maker from the observed preference information without assuming a specific parametric functional form. Therefore, this paper proposes a nonparametric numerical approach to approximate PWF, combining parameter approach and nonparametric elicitation approach. The collected preference points come from two parts. In the first part, experimental preference points (which are defined in Definition 4) are obtained by eliciting experiment or questionnaire survey. In the second part, computation preference points (which are also defined in Definition 4) are obtained by utilizing prelec's two-parameter model to infer several preference points related to the elicitation experimental preference points.

2. Preliminaries

2.1. Preference relation

Let E and V be the sets whose elements are interpreted as event consequences (such as monetary outcomes of a lottery or gamble) of the world and the utility of corresponding to event consequences, respectively; P and W be the sets whose elements are the objective probabilities of events and the psychological probabilities of events for the decision-maker, $P = [0,1]$ and $W = [0,1]$. R is the set of real numbers. φ is denoted as the impossible event, ε as the possible event and G as a certain event, such that $\varphi, \varepsilon \in E$, and the probabilities are $p(G) = 1$, $p(\varphi) = 0$ and $0 < p(\varepsilon) < 1$. A PWF w is a mapping that assigns a number between 0 and 1 to the probability of each event in E. The psychological probabilities are $w(p(G)) = 1$, $w(p(\varphi)) = 0$ and $0 < w(p(\varepsilon)) < 1$. For simplicity of exposition, we assume a finite set of monetary outcomes, such as $E = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$. The monetary outcomes are ordered from small to big, i.e., $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{n-1} < \varepsilon_n$. A prospect that comprises some monetary outcomes is assumed as a finite probability distribution over the set E and represented by $Q = (\varepsilon_1, p_1; \ \varepsilon_2, p_2; \ \cdots; \ \varepsilon_n, p_n)$, which implies that probability $p_i \ (i = 1, 2, \cdots, n)$ is assigned to monetary outcomes $\varepsilon_i \in E$, for $i = 1, 2, \cdots, n$. The utility of the prospect Q is denoted by $U(Q)$, the utility of the monetary outcomes $\varepsilon_i \ (i = 1, 2, \cdots, n)$ is denoted by $v(\varepsilon_i)(i = 1, 2, \cdots, n)$.

Only three types of prospects are considered in this paper, namely, certain prospects $(\varepsilon, 1)$, two-outcome prospects with one zero outcome $(\varepsilon, p; 0, 1-p)$ simplified as (ε, p) and three-outcome prospects with one zero outcome $(\varepsilon, p; \tilde{\varepsilon}, q; 0, 1-p-q)$, also can be abbreviated as $(\varepsilon, p; \tilde{\varepsilon}, q)$, and known as binary prospect. For convenience, a definition on preference relation is first defined in the following context.

Definition 1: The symbol $\precsim$ represents weak preference relation or "at most as good as", $\prec$ represents strict preference relation and $\sim$ denotes equivalence or indifference to preference relation.

Note that, when some prospects $q_i, q_j \in Q$, if $q_i \precsim q_j$, then that decision-maker weakly prefers to q_j over q_i; if $q_i \prec q_j$, then that decision-maker strictly prefers to q_j over q_i; and if $q_i \sim q_j$, which shows that decision-maker exhibits equivalence or indifference to q_i and q_j. For instance, three two-consequence prospects are exhibited as follows:

$$q_1 = (\varepsilon_1, p_1; \tilde{\varepsilon}_1, 1-p_1),$$
$$q_2 = (\varepsilon_2, p_2; \tilde{\varepsilon}_2, 1-p_2),$$
$$q_3 = (\varepsilon_3, p_3; \tilde{\varepsilon}_3, 1-p_3), \tag{1}$$

where $q_1, q_2, q_3 \in Q$, $\varepsilon_1, \tilde{\varepsilon}_1, \varepsilon_2, \tilde{\varepsilon}_2, \varepsilon_3, \tilde{\varepsilon}_3 \in E$, and $p_1, p_2, p_3 \in P$. In decision-making theory, the utility of the decision-maker for the three prospects are solved by the following equation:

$$U(q_i) = v(\varepsilon_i) w(p_i) + v(\tilde{\varepsilon}_i) w(1-p_i), \ i = 1, 2, 3. \tag{2}$$

In Eq. (2), v is a mapping from the set of monetary outcomes E into the set of real numbers R. Its properties will be displayed in Property 1, such that for all monetary outcomes $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in E$, and the preference relation $\precsim$ must satisfy transitivity:

$$\varepsilon_1 \precsim \varepsilon_2, \varepsilon_2 \precsim \varepsilon_3 \Rightarrow \varepsilon_1 \precsim \varepsilon_3 \tag{3}$$

and the symbol $\precsim$ satisfies monotonicity

$$\varepsilon_1 \precsim \varepsilon_2 \Leftrightarrow v(\varepsilon_1) \precsim v(\varepsilon_2). \tag{4}$$

Eqs (3) and (4) obtain the transitive monotonicity as follows:

$$\varepsilon_1 \precsim \varepsilon_2, \varepsilon_2 \precsim \varepsilon_3 \Rightarrow v(\varepsilon_1) \precsim v(\varepsilon_3). \tag{5}$$

Obviously, the preference relations such as $\prec$ and $\sim$ also satisfy the above properties.

At the same time, w is mapped from the set of event probabilities P into the set of probability weights W. Its properties will be described again in Property 2. For preference relations, it must satisfy transitivity:

$$w(p_1) \precsim w(p_2), \ w(p_2) \precsim w(p_3) \Rightarrow w(p_1) \precsim w(p_3). \tag{6}$$

Of course, the preference relations such as $\prec$ and $\sim$ are also the same as $\precsim$.

2.2. Decision-making theories

Several researchers pay attention to PT (Kahneman & Tversky, 1979) and its extension form CPT (Tversky & Kahneman, 1992). PT as a new behavior under risk decision model was proposed, which not only affected the psychology field, but also synthesized psychology and other subjects, such as economics, philosophy and risk management. PT distinguishes two phases in the choice process: (I) the framing phase, in which the decision-maker constructs a representation of the acts, reflecting on the value function, and (II) the valuation phase, in which the decision-maker assesses the probability of each prospect and chooses accordingly. CPT is the extension of PT, and is a widely used descriptive model of decision-making under risk. They are both made up of two parts to express the utility of the decision-maker. This classification is attractive because people generally accepted a normative theory and its framework. Expect utility theory (EUT, Von Neumann & Morgenstern, 1944) is one of the earliest decision-making theories under risk.

EUT offered a descriptive model of economic behavior for decision-making under risk in which events probabilities are given, and the utility function is a single measure to capture all aspects of risk attitude. Its utility of prospect Q is expressed as follows:

$$U_{EU}(Q) = EP = \varepsilon_1 p_1 + \varepsilon_2 p_2 + \cdots + \varepsilon_n p_n. \tag{7}$$

Kahneman and Tversky (1979) proposed PT as an alternative model. The utility of prospect Q can be represented symbolically as:

$$U_{PT}(Q) = V(E)W(P) = \sum_{i=1}^{n} v(\varepsilon_i) w(p_i), \tag{8}$$

where $w(p_i)(i = 1, 2, \cdots, n)$ is the probability weight of the decision-maker, which is non-linear, has an inverse S-shape, and is regressive. Through treating probability non-linearly, Tversky and Kahneman (1992) attempted to capture such deviations from objective probabilities formally by absorbing rank- and sign-dependent utility theories (Luce & Fishburn, 1991) and cumulative function and improving prospect theories, as well as proposed CPT with a psychological interpretation for overweighting and underweighting. Using CPT, the utility of prospect Q can be represented as

$$U_{CPT}(Q) = V(E)W(\tilde{P}) = \sum_{i=1}^{h} v(\varepsilon_i) \tilde{w}^-(p_i) + \sum_{i=h+1}^{n} v(\varepsilon_i) \tilde{w}^-(p_i), \tag{9}$$

where, the reference point is r, and $\varepsilon_1 \prec \varepsilon_2 \prec \cdots \prec \varepsilon_h \prec r \prec \varepsilon_{h+1} \prec \cdots \prec \varepsilon_n$. $\tilde{w}^-(p)$ and $\tilde{w}^+(p)$ are the cumulative probability distribution function of the decision-making result. More specifically, $\tilde{w}^-(p)$ represents the probability weighting of the positive prospect.

$$\tilde{w}^-(p_i) = w\left(\sum_{k=i}^{h} p_k \right) - w\left(\sum_{k=i+1}^{h} p_k \right) \quad k = 1, 2, \cdots, h \tag{10}$$

and $\tilde{w}^+(p)$ represents the probability weighting of the negative prospect.

$$\tilde{w}^+(p_j) = w\left(\sum_{k=j}^{n} p_k \right) - w\left(\sum_{k=j+1}^{n} p_k \right) \quad k = h+1, \cdots, n. \tag{11}$$

In the two decision-making theories, PT introduced a nonlinear weighting function (such as exponential function $w(p) = e^p$ and power function $w(p) = p^\beta$) to investigate the decision-making behavior of a normal person that integrates the value $v(\varepsilon)$ of personal feeling factors into the decision-making behavior analysis ($v(\varepsilon)w(p)$). CPT considers the distinguish between the PWF and the decision weight $w(p)$ based on reference points as useful. $\tilde{w}^-(p)$ and $\tilde{w}^+(p)$ are the cumulative probability distribution functions of the decision-making result, and $v(\varepsilon)$ distinguishes the gains and losses based on reference point r.

In this paper, we propose a nonparametric numerical approach to approximate PWF under the PT and CPT frameworks, and the approximated nonparametric numerical model is based on the elicitation experimental preference points, which are the response to the risk preferences of the decision-maker. Moreover, it is easy to know that different decision-making theories have different forms of PWF, different PWF models also reflect different risk attitudes of the decision-maker, and different risk attitudes are the response to the different risk preferences. Thus, investigating preference relations under different decision-making theories is an important part of proposing a nonparametric numerical approach.

U is a combination of utility function v and PWF w. For several prospects $q_1, q_2, q_3 \in Q$, the preference relations satisfy the following some properties:

$$q_1 \precsim q_2 \Leftrightarrow U(q_1) \precsim U(q_2); \tag{12}$$

$$q_1 \sim q_2 \Leftrightarrow U(q_1) \sim U(q_2); \tag{13}$$

$$U(q_1) \sim U(q_2), U(q_2) \sim U(q_3) \Rightarrow U(q_1) \sim U(q_3). \tag{14}$$

For instance, under different decision-making theories, two simple two-outcome prospects, that is $q_1 = (\$100, 0.5; \$200, 0.5), q_2 = (\$200, 0.6; \$75, 0.4)$ and a certainty prospect $q_3 = (\$140, 1.0)$ are considered. q_1 accounted for a prospect that a game offers a 50% chance to win \$100 and a 50% chance to win \$200. The interpretations of two-consequence prospects q_2 and q_3 are similar to q_1. A comparison of the preference relation of the decision-maker for the three prospects indicates the following:

(1) EU theory is used to solve the utility of three prospects,

$$U_{EU}(q_1) = 150, U_{EU}(q_2) = 150 \text{ and } U_{EU}(q_3) = 140.$$

Based on EU theory, we obtain the preference relation.

$$U_{EU}(q_3) \prec U_{EU}(q_1) \sim U_{EU}(q_2). \tag{15}$$

(2) PT is used to solve the utility of three prospects,

$$U_{EU}(q_1) = 100 + 100 \cdot w(0.5),$$

$$U_{EU}(q_2) = 75 + 125 \cdot w(0.6),$$

$$U_{EU}(q_3) = 140.$$

Risk attitude is divided into two parts because the decision-maker has different risk attitudes in case of uncertainty.

I. Assume that the decision makers are risk-seeking, $w(0.5) = 0.52$ and $w(0.6) = 0.64$,

$U_{PT}(q_1) = 152$, $U_{PT}(q_2) = 155$ and $U_{PT}(q_3) = 140$.

According to PT, the preference relation is exhibited as follows:

$$U_{PT}(q_3) \prec U_{PT}(q_1) \prec U_{PT}(q_2). \tag{16}$$

II. Assume that the decision maker is risk-averse, $w(0.5) = 0.40$ and $w(0.6) = 0.5$,

$U_{PT}(q_1) = 140$, $U_{PT}(q_2) = 137.5$ and $U_{PT}(q_3) = 140$.

According to PT, the preference relation is exhibited as follows:

$$U_{PT}(q_2) \prec U_{PT}(q_3) \sim U_{PT}(q_1). \tag{17}$$

(3) The cumulative probability of CPT is the same as PT for two-consequence prospects. Thus, the results of using CPT to solve the utility of three prospects are also (15) and (16). If a multiple-consequence prospect exists, Eqs (3), (4) and (5) could be utilized to solve the preference relation of this prospect. However, the cumulative PWF also depends on the risk attitudes of the decision-maker.

EU theory can be treated as a risk-neutral risk attitude for decision makers. PT and CPT consider risk attitudes not only as risk-neutral, but also as risk-seeking and risk-aversion. Eqs (15)–(17) showed that finding the PWF based on risk attitude plays an important role in decision making under risk and uncertainty. To understand the two main components of CPT, the value function $v(\varepsilon)$ and the PWF $w(p)$ are redefined in Sections 2.3 and 2.4.

2.3. Utility function

Property 1: Value function $v(\varepsilon)$ satisfies the following properties:

(1) Risk aversion for gains.

(2) Risk seeking for losses.

(3) People are more sensitive to losses than to gains of the same magnitude.

Toubia et al. (2013) used the following value function to apply their approach to a version of CPT:

$$v(\varepsilon, \sigma) = \begin{cases} \varepsilon^{\sigma} & \text{for } \varepsilon > 0, \\ -\lambda(-\varepsilon)^{\sigma} & \text{for } \varepsilon < 0, \end{cases} \tag{18}$$

where ε is a variable that represents the event consequence. σ captures the curvature of value function, and λ is loss aversion.

Toubia et al. (2013) used a different curvature of the value function for the sign-dependent CPT.

$$v(\varepsilon, \sigma^+, \sigma)^- = \begin{cases} \varepsilon^{\sigma^+} & \text{for } \varepsilon > 0, \\ -\lambda(-\varepsilon)^{\sigma^-} & \text{for } \varepsilon < 0, \end{cases} \tag{19}$$

where σ^+ and σ^- are free parameters that vary between 0 and 1 and modulate the curvature of the subjective value functions (i.e., $0 < \sigma^+ \leq \sigma^- < 1$). When the parameter $\sigma^+ = \sigma^-$, Eq. (18) is equal to Eq. (19). Moreover, parameter λ specifies loss aversion (see Schmidt and Zank (2005), Abdellaoui et al. (2008), and so on), with larger values expressing larger loss aversion (i.e., $\lambda > 1$).
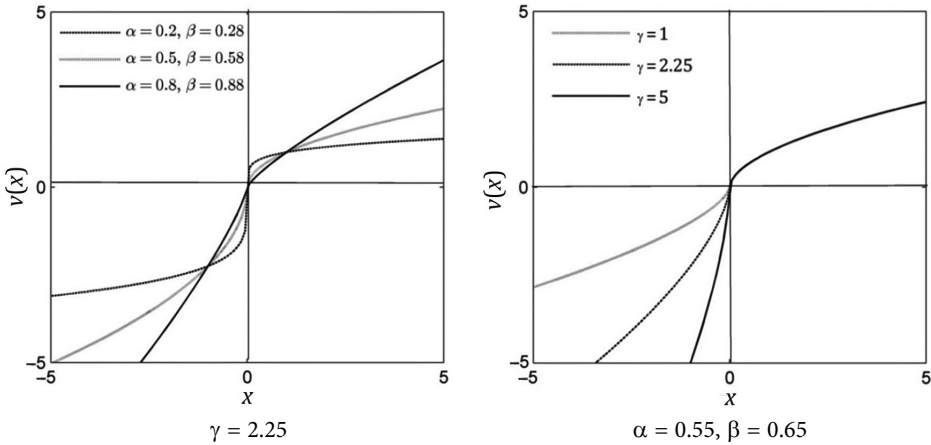
Figure 1. Value function $v(x)$ for different values of α, β and γ. The value function is concave over gains and convex over losses; as the parameter γ is larger, the losses are steeper

In summary, decision-maker is risk-seeking for small probability gains and large probability losses, risk-aversion for small probability losses and large probability gains. The slope is steeper for losses than for gains. A value function that satisfies these properties is displayed in Figure 1.

2.4. Probability weight function

Property 2: PWF $w(p)$ satisfies the following properties:

(1) $w(p)$ is a strict increasing function from $\left[0,1\right]$ to $\left[0,1\right]$ with $w(0)=0$ and $w(1)=1$. The strict increasing indicates that for all p_1, $p_2 \in (0,1)$, there is $p_1 < p_2$, such that $w(p_1) < w(p_2)$.

(2) Overweighting for small values of p implies that a lower interval $\left[0,q\right]$ has a greater effect on decision makers than an intermediate interval $\left[p,q+p\right]$, provided that $q + p$ is bounded away from 1.

(3) Underweighting for large values of p, implies that a higher interval $\left[1-q,1\right]$ has a greater effect on decision makers than an intermediate interval $\left[p,q+p\right]$, provided that p is bounded away from 0.

(4) $w(p)$ for losses is more pronounced than that for gains, and $w(p)$ for large magnitude monetary outcomes is more pronounced than that for small magnitude monetary outcomes.

(5) Subadditivity exists at the boundaries of interval $\left[0,1\right]$. There is lower subadditivity for small probabilities, this is that if $w(p) > p$, then $w(rp) > rw(p)$ for $0 < r < 1$; and upper subadditicity for large probability, i.e., if $w(p) < p$, then $w(rp) < rw(p)$ for $0 < r < 1$.

(6) Subproportionality for all $0 < p, q, r \le 1$. If $w(p)v(\varepsilon_1) = w(pq)v(\varepsilon_2)$ implies $w(pr)v(\varepsilon_1) \le w(pqr)v(\varepsilon_2)$; hence, $\dfrac{w(pq)}{w(p)} \le \dfrac{w(pqr)}{w(pr)}$.

PWF is an important component and models the distortion of probability, as well as characterizes the psychophysics of chance. PWF was initially developed in the last century (Hong & Waller, 1986; Lattimore et al., 1992; Wu & Gonzalez, 1996, and so on). This PWF is the psychological reaction of the decision-maker, and could reflect and predict the risk preferences of the decision-maker. Considering the important role of PWF in decision-making analysis, some assumptions for the functional forms have been presented by testing these functional forms for the PWF in what can be described as a "goodness of fit contest" based on different choices. For instance, Tversky and Kahneman (1992) and Prelec (1998) estimated PWF by proposing different one-parameter models (Figure 2). At the same time, Tversky and Fox (1995) and Prelec (1998) offered two-parameter models to approximate PWF (Figure 3). Moreover, Diecidue et al. (2009) proposed switch-power PWF with the three-parameter model (Figure 4).
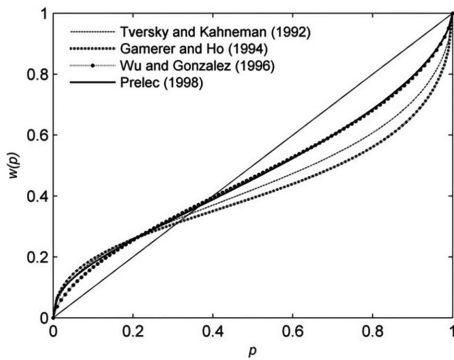
Figure 2. One-Parameter PWF models were established by Tversky and Kahneman (1992), Camerer and Ho (1994) and Wu and Gonzalez (1996) using $w(p) = \left(p^{\beta} \middle/ \left(p^{\beta} + \left(1 - p^{\beta}\right)\right)\right)^{1/\beta}$ with taking different parameter values $\beta = 0.61, 0.56, 0.71$, and Prelec (1998) using $w(p) = \exp\left(-\left(-\ln p\right)^{\alpha}\right)$ with $\alpha = 0.65$
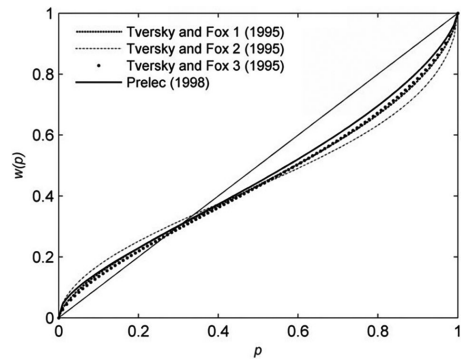
Figure 3. Two-Parameter PWF models were established by Tversky and Fox (1995) by using $w(p) = \mu p^{\eta} \middle/ \left(\mu p^{\eta} + \left(1 - p\right)^{\eta}\right)$ with different parameters $\eta_1 = 0.69$, $\mu_1 = 0.77$; $\eta_2 = 0.69$, $\mu_2 = 0.76$; $\eta_3 = 0.72$, $\mu_3 = 0.76$, and Prelec (1998) using $w(p) = \exp\left(-\beta\left(-\ln p\right)^{\alpha}\right)$ with $\alpha = 0.71$, $\beta = 1.05$
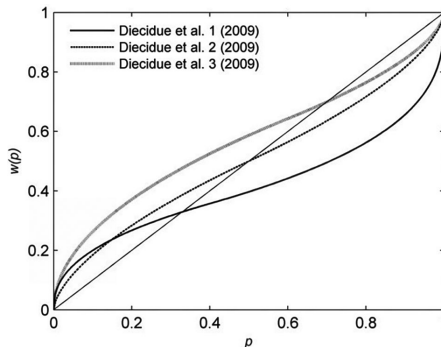
Figure 4. Three-Parameter PWF models were established by Diecidue et al. (2009) by using $w(p) = cp^{a}$ for $p \le \hat{p}$ and $w(p) = 1 - d\left(1 - p\right)^{b}$ for $p > \hat{p}$, in where, $c = \hat{p}^{1-a} b \middle/ \left(b\hat{p} + a\left(1 - \hat{p}\right)\right)$ and $d = a\left(1 - \hat{p}\right)^{1-b} \middle/ \left(b\hat{p} + a\left(1 - \hat{p}\right)\right)$. The parameters a, b and $\hat{p}$ are 0.42, 0.35 and 0.40; 0.62, 0.62 and 0.50; 0.50, 0.60 and 0.60, respectively

Overall, the PWF with the parameter form is proposed by the separation of probability weighted and value function, and probability weighted is the psychological reaction of decision-maker under risk and uncertainty. This paper confirms that PWF could reflect and predict the risk preferences of decision-maker, and their curves were also considered as the risk preference curves of decision-maker. When the curve is above the diagonal, decision-maker is risk-seeking; when the curve is under the diagonal, decision-maker is risk-aversion. According to these characteristics, researchers could judge the risk preferences of decision-maker. To estimate the risk preferences of decision-maker by using parameter models, the parameters of the models could be determined by least squares or maximum likelihood method with curve fitting of several preference points, which could be collected by eliciting experiment or questionnaire survey. PWFs with parameter forms are flexible for the number of requirements of preference points in estimating the risk preferences of decision-maker according to the number of parameters. However, the disadvantages of parametric estimation are its functional form which makes more assumptions, when the parameters are fixed, the assumed functional form has also been determined, if the true functional form is different from the assumed functional form, then it cannot determine whether the measurement are driven by the data. Thus, the fixed model could not flexibly reflect and predict the preferences of decision makers. To circumvent this problem, some researchers, Such as Wakker and Deneffe (1996), Gonzalez and Wu (1999), Abdellaoui (2000), Bleichrodt and Pinto (2000), Abdellaoui et al. (2005), Blavatskyy (2006), Abdellaoui et al. (2007), Booij et al. (2009) and Van Houtven et al. (2011) proposed parameter-free elicitation method to elicit PWF under different theoretic frameworks and obtained good results.

More specifically, PWF with parameter-free form is approximated by using several elicitation methods such as certainty equivalent method (Krzysztofowicz (1983) and Farquhar (1984)), probability equivalence method (Hershey & Schoemaker, 1985) or the trade-off method (Wakker & Deneffe, 1996) to elicit utility function and PWF with making no prior assumptions about the functional forms. This method is used in the experimental survey to elicit individual and aggregate PWF both in the gain and loss domain. The main advantages of this method are that it allows researchers to infer the psychological weights from the observed experimental data, and ensures parameter-free estimation with fewer assumptions, ensuring that the results remain close to the experiment data (Gonzalez & Wu, 1999). Abdellaoui (2000) used two successive steps (i.e., constructing a standard sequence of outcomes, and determining a corresponding standard sequence of probabilities) to experimentally elicit the PWF. The elicited weighting functions satisfy subadditivity near the boundary of interval [0, 1]. Furthermore, the result showed that the obtained shape and properties are consistent with the above parameter PWF estimations. Bleichrodt and Pinto (2000) also presented a trade-off method to elicit PWF and applied this method in medical decision analysis. The conclusion that probability weighting is robust both at the aggregate and individual subject level is obtained. Moreover, PWF is inverse S-shaped, which is consistent with Gonzalez and Wu (1999).

Based on the previous properties of PWF defined in Property 2, several new properties of PWF are redefined in this paper as follows:

Definition 2: Based on preference relation and the existing properties of PWF, a piecewise PWF is redefined as follows:

$$w(p) = \begin{cases} 0 & \text{for } p = 0, \\ N(p) & \text{for } p \in (0, 1), \\ 1 & \text{for } p = 1, \end{cases} \quad (20)$$

where $N(p)$ is a function and satisfies the following properties:

(1) $N(p)$ is a weakly increasing function from $(0, 1)$ to $(0, 1)$. The weak increase indicates that for almost all $p_1, p_2 \in (0,1)$, there is $p_1 < p_2$, such that $N(p_1) < N(p_2)$.

(2) Overweighting for small values of p, implies that a lower interval $(0, q]$ has a greater effect on the decision-maker than an intermediate interval $[p, q+p]$, provided that $q + p$ is bounded away from 1.

(3) Underweighting for large values of p, implies that a higher interval $[1-q, 1)$ has a greater effect on decision-maker than an intermediate interval $[p, q+p]$, provided that p is bounded away from 0.

(4) $N(p)$ for losses is more pronounced than that for gains; and $N(p)$ for large magnitude monetary outcomes is more pronounced than that for small magnitude monetary outcomes.

(5) Subadditivity at the boundaries of interval $(0, 1)$. There is lower subadditivity for small probabilities, this is that if $N(p) > p$, then $N(rp) > rN(p)$ for $0 < r < 1$; and upper subadditicity for big probability, i.e., if $N(p) < p$, then $N(rp) < rN(p)$ for $0 < r < 1$.

(6) Subproportionality. for all $0 < p, q, r < 1$. if $N(p)v(\varepsilon_1) = N(pq)v(\varepsilon_2)$ implies $N(pr)v(\varepsilon_1) \leq N(pqr)v(\varepsilon_2)$; hence, $\dfrac{N(pq)}{N(p)} \leq \dfrac{N(pqr)}{N(pr)}$.

In Definition 2, $N(p)$ is a nonlinear function without giving a specific parametric functional form, $w(p)$ is a piecewise nonlinear function[1] and discontinuous at points $p = 0$ and $p = 1$ ($\lim_{p \to 0} w(p) \neq 0$, $\lim_{p \to 1} w(p) \neq 1$). This condition is due to the fact that people do not consider very small probability event ($p \to 0$) as impossible event (i.e., $\lim_{p \to 0} w(p) \neq 0$), such as the lottery; people also do not consider very big probability event ($p \to 1$) as sure event (i.e., $\lim_{p \to 1} w(p) \neq 1$), such as insurance.

Definition 2 defined a weakly increasing function in the interval $(0, 1)$ because the risk preferences of the decision-maker change according to the environment. PWF does not always exhibit strict increase in the actual decision-making. The empirical studies by Gonzalez and Wu (1999) indicated the result. More specifically, Gonzalez and Wu (1999) shows the

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[1] Chateauneuf et al. (2007) used a piecewise probability weighting function model which is linear and (possibly) discontinuous at 0 and at 1. Its form is given as follows:

$$w(p) = \begin{cases} 0 & \text{for } p = 0 \\ \alpha p + \beta & \text{for } 0 < p < 1 \\ 1 & \text{for } p = 1 \end{cases}$$

with $0 \leq \beta < 1$ and $0 < \alpha \leq 1 - \beta$. At the same time, the properties of small probabilities overweighting and large probabilities underweighting lead to the discontinuity at the endpoints.

best fitting linear in log odds PWF for aggregate data, in which, the empirical data, that is, probability 0.5 > 0.4, but $w(0.5) < w(0.4)$, demonstrated the change of risk preferences. Moreover, Gonzalez and Wu (1999) also shows that the empirical data for subjects 1, 2, 3, 4, 5, 7 and 8 do not always satisfy the property of a strict increase. The redefined PWF still satisfies the properties such as an inverse s-shape; fourfold pattern of risk preferences (overweighting for small probabilities, underweighting for moderate and big probabilities); subadditivity at the boundaries of the unit interval; Sub-proportionality; and so on.

3. Nonparametric numerical approach

This section proposes a nonparametric numerical approach to approximate PWF. The nonparametric numerical approach consists of four subsections. In the first subsection, a numerical approximation approach, namely, Newton interpolation, is introduced to build a PWF model to fully approximate several experimental data, and its function and properties are given. In the second subsection, an elicitation certainty equivalent method is redefined to elicit PWF for collecting the experimental data. Subsection 3.3 takes into account using the Newton interpolation approach to interpolate the elicitation PWF data (experimental preference points, EPPs are defined in Subsection 3.2). Subsection 3.4 offers an improved approach to increase several new PWF data (computational preference points, CPPs defined in Subsection 3.2) by combining the parameter PWF model (i.e., Prelec's two-parameter PWF model) with the elicitation PWF data.

3.1. Newton interpolation approach

Interpolation method is a numerical computation method, which are widely used in every field. This method is developed for the interpolation of a given set of data points in a plane and to fit a smooth curve to these points. This method is devised in such a way that the resultant curve will pass through the given points and will appear smooth and natural. This approach is based on a piecewise function composed of a set of polynomials, and the slope of the curve is determined at each given points locally. Newton interpolation is one of interpolation approaches. Its advantage is that it – *CVV0increases or updates a new knowledge for each new increasing or updating data point to improve the response to the interpolated function. In this paper, the interpolated function is PWF, which uses the values of PWF at the finite points to estimate the values of PWF at the other points. Accurate approximate results are obtained only by increasing the number of data points, i.e., increasing the density of nodes. Subsection 3.4 proposes a nonparametric numerical approach to increase several computation preference points by using Prelec's parameter model in order to build a better performance's approximation model than the numerical model Subsection 3.3, which is developed only by using Newton interpolation to interpolate several elicitation experimental preference points. To introduce the detail of Newton interpolation approach, the following lemma is introduced as follows:

Suppose the data sets can be expressed as $\left(x_i, f\left(x_i\right)\right)$ $\left(i = 0, 1, \cdots, k\right)$, where $f\left(x_i\right)$ is the value of $f\left(x\right)$ corresponding to x_i $\left(i = 0, 1, \cdots, k\right)$.

Lemma 1: (The definition of difference quotient) It assumes that there are $k + 1$ nodes $\left(x_i, f\left(x_i\right)\right)$ $\left(i = 0, 1, \cdots, k\right)$ in the interval $\left[a, b\right]$. The first-order difference quotient of nodes $\left(x_i, f\left(x_i\right)\right)$ and $\left(x_j, f\left(x_j\right)\right)$ are defined by

$$f\left[x_i, x_j\right] = \frac{f\left(x_i\right) - f\left(x_j\right)}{x_i - x_j}, \tag{21}$$

where x_i and x_j $\left(i, j \in 0, 1, \cdots, k\right)$ are different.

The difference quotient of the $k - 1$-order difference quotient is known as k-order difference quotient:

$$f\left[x_0, x_1, \cdots, x_{k-1}, x_k\right] = \sum_{i=0}^{k}\left(\prod_{\substack{j=0 \\ j \neq i}}^{k} \frac{f\left(x_i\right)}{\left(x_i - x_j\right)}\right), \tag{22}$$

where $x_0, x_1, \dots x_{k-1}$ and x_k are both different. In particular, zero-order difference quotient is denoted as follows:

$$f\left[x_k\right] = f\left(x_k\right). \tag{23}$$

Lemma 2: The k-order Newton interpolation polynomial can be expressed as

$$N_k\left(x\right) = \sum_{i=0}^{k-1}\left(a_i \prod_{j=1}^{i}\left(x - x_j\right)\right), \tag{24}$$

where a_k $\left(i = 0, 1, \cdots, k-1\right)$ are the coefficient of Newton interpolation polynomial, which is obtained by the interpolation conditions:

$$N_k\left(x_i\right) = f\left(x_i\right) \quad \left(i = 0, 1, \cdots, k-1\right). \tag{25}$$

Using difference quotients to express the coefficient a_k of Newton interpolation polynomial $N_k(x)$, applying LEMMA 1, these coefficients could be solved as follows:

$$a_0 = f\left(x_0\right) = f\left[x_0\right], \tag{26}$$

$$a_l = \sum_{i=0}^{l}\prod_{\substack{j=0 \\ j \neq i}}^{l} \frac{f\left(x_i\right)}{\left(x_i - x_j\right)} = f\left[x_0, x_1, \cdots, x_l\right], \quad l = 1, 2, \cdots, k-1.$$

The k-order Newton interpolation polynomial can also be expressed as

$$N_k(x) = \sum_{i=0}^{k-1}\left(a_i \prod_{j=1}^{i}\left(x - x_j\right)\right). \tag{27}$$

This polynomial has degree $\leq k$ and has the property $N_k\left(x_i\right) = f\left(x_i\right)$ as required. To display the properties of Newton interpolation coefficients a_i $\left(i = 0, 1, \cdots, k-1\right)$ and Newton interpolation polynomial $N(x)$, Properties 3 and 4 are presented as follows:

Property 3: The coefficients of Newton interpolation polynomial, a_i $\left(i = 0, 1, \cdots, k-1\right)$, satisfy the following property.

$$a_0 = f(x_0),$$

$$a_{r+1} = a_r \sum_{i=0}^{r} \frac{1}{(x_i - x_{r+1})} + \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{f(x_{r+1})}{(x_{r+1} - x_j)}, \tag{28}$$

where $r = 0, 1, \cdots, k - 2$.

Proof: From Eq. (34), we know that

$$a_r = \sum_{i=0}^{r} \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{f(x_i)}{(x_i - x_j)} \quad \text{and} \quad a_{r+1} = \sum_{i=0}^{r+1} \prod_{\substack{j=0 \\ j \neq i}}^{r+1} \frac{f(x_i)}{(x_i - x_j)}.$$

Therefore, expanding a_{r+1}, the following expression can be obtained

$$a_{r+1} = \sum_{i=0}^{r+1} \prod_{\substack{j=0 \\ j \neq i}}^{r+1} \frac{f(x_i)}{(x_i - x_j)} = \sum_{i=0}^{r} \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{f(x_i)}{(x_i - x_j)} \cdot \sum_{i=0}^{r} \frac{1}{(x_i - x_{r+1})} + \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{f(x_{r+1})}{(x_{r+1} - x_j)} =$$

$$a_r \cdot \sum_{i=0}^{r} \frac{1}{(x_i - x_{r+1})} + \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{f(x_{r+1})}{(x_{r+1} - x_j)}.$$

When $r = 0, 1 \cdots, k - 2$, Eq. (28) stands, and thus, Eq. (28) is proven.

Property 4: The Newton interpolation polynomial, $N(x)$, is a unique existence.

Proof: (Existence) Through the structural process of Newton interpolation polynomial $N(x)$, we know that the polynomial $N(x)$ exists.

(Unique) If two such polynomials $N(x)$ and $P(x)$ exist, which are obtained by the $k + 1$ nodes, then

$$\text{degree}(N(x)) = \text{degree}(P(x)) = k,$$

$$\text{degree}(N(x) - P(x)) \leq k, \tag{29}$$

$$N(x_i) - P(x_i) = 0 \quad (i = 0, 1, \cdots, k).$$

Thus the result would have to be identically zero, and thus, a corollary is $N(x) \equiv P(x)$.

Although the coefficients of the computed Newton interpolation polynomial are relatively complex, this problem is easy to solve with the development of computer technology. Moreover, Property 3 provides a recurrence formula to facilitate convenient calculation. Property 4 explains Newton interpolation polynomial for the existing and unique interpolation nodes. These properties establish a solid foundation for the proposal of a nonparametric numerical approach.

3.2. Eliciting PWF

Using Newton interpolation approach to approximate PWF, several interpolation nodes indicate that the objective probability that corresponds to psychological probability should be obtained. To collect several experimental data that could reflect the risk preferences of decision-maker, a certainty equivalent method is defined as follows:

Definition 3: Assume a two-consequence prospect $q = (\varepsilon, p; \tilde{\varepsilon}, 1-p)^2$ and a certain prospect $q' = (c, 1)$, where, $\varepsilon, \tilde{\varepsilon} > 0$, $|\varepsilon| > |\tilde{\varepsilon}|$ and decision makers think $q \sim q'$. The probability weight that corresponds to the probability p is solved by using the following transformation:

$$\widehat{w}(p) = \left| \frac{c - \tilde{\varepsilon}}{\varepsilon - \tilde{\varepsilon}} \right|. \tag{30}$$

This method is known as the certainty equivalent (C-E) method.

C-E method (Farquhar, 1984) allows the elicitation of a two-outcome prospect by some sure prospects. The researcher provides a two-outcome prospect and some sure prospects to elicit the choice of the decision-maker under certain circumstances in order to obtain a sure prospect which is equivalent to the two-outcome prospect for the decision-maker. In other words, the obtained sure prospect and the two-outcome prospect are indifference to the risk preferences of the decision-maker.

The use of the C-E method to elicit probability weighting could obtain the preferences information of the decision-maker by means of probability $\widehat{w}(p)$, which the decision-maker considers the psychological probability of objective probability p. To express the relation between the objective probability and psychological probability, Definition 4 is proposed as follows:

Definition 4: The objective probabilities $p_i (i = 1, 2, \cdots, n)$ and they corresponding to the elicitation probability weights $\widehat{w}(p_i)(i = 1, 2, \cdots, n)$ constitute the two-dimensional array $(p_i, \widehat{w}(p_i))(i = 1, 2, \cdots, n)$, known as preference points (PPs). The curve of $\widehat{w}(p)$ obtained by fitting or others methods such as parameter-free elicitation method to approximate these preference points $(p_i, \widehat{w}(p_i))(i = 1, 2, \cdots, n)$ is known as preference curve. They reflect and predict the risk attitudes of decision makers under risk and uncertainty.

(1) The objective probabilities $p_i (i = 1, 2, \cdots, n)$ are the probabilities of monetary outcomes $\varepsilon_i (i = 1, 2, \cdots, n)$.

(2) The probability weights $\widehat{w}(p_i) (i = 1, 2, \cdots, n)$ are the psychological reactions of decision makers for monetary outcomes $\varepsilon_i (i = 1, 2, \cdots, n)$, they are obtained by using C-E to elicit PWF, and they are significantly influenced by the uncertainty of events and the risk attitudes of decision makers.

(3) PPs $(p_i, \widehat{w}(p_i))(i = 1, 2, \cdots, n)$ collected through experiment or questionnaire survey are known as the experimental preference points (EPPs).

(4) PPs $(p_j, \widehat{w}(p_j))(j = 1, 2, \cdots, l)$ collected through computation are known as the computational preference points (CPPs).

Definition 4 defines three terms that, reflect the risk preferences of decision-maker, that are, preference points (PPs), experimental preference points (EPPs) and computational preference points (CPPs), in order to facilitate the expression of Subsections 3.3 and 3.4. Because Subsection 3.3 utilizes the elicitation EPPs to approximate PWF, yet Subsection 3.4 but utilizes only the elicitation EPPs, as well as increases several CPPs that are related to the EPPs, to approximate PWF.

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[2] The prospect $q = (\varepsilon, p; \tilde{\varepsilon}, 1-p)$ represents a gamble offering a p chance to obtain the monetary outcomes ε and a $1 - p$ chance to obtain the monetary outcomes $\tilde{\varepsilon}$.

3.3. Newton Interpolation to Approximate PWF

To build a numerical PWF model by using Newton interpolation[3], several elicited PPs (i.e., EPPs), which are collected by using C-E method[4] to process several observed experimental data from experiment or questionnaire survey, are considered as interpolation nodes. The form of the approximation PWF model is similar to Eq. (27).

It assumes that an elicitation experiment or questionnaire survey[5] is conducted to collect several experimental data, such as those shown in Tables 2 and 6 in Section 4. The collected experimental data are processed by Eq. (30) defined in Definition 3 to transform monetary outcomes ε_i into probability weights $\widehat{w}(p_i)$ with probabilities p_i, where $i = 1, 2, \cdots, n$. For convenience our expression, it defines the set of EPPs by S_1.

$$\left(p_i, \widehat{w}(p_i)\right) \in S_1, \quad i = 1, 2, \cdots, n. \tag{31}$$

Set S_1 is considered as the elicitation PWF data set, which is the response to the risk preferences of the decision-maker. When these experimental data represent the risk preferences of most people (aggregate level), using Newton interpolation could build a model to reflect the risk preferences of most people. At the same time, if they represent individual data (individual level), then using Newton interpolation could also build such a model that reflects the individual risk preferences. It is necessary to state that the built model could well reflect and predict the risk preferences of decision-maker as long as the experimental data are high-quality[6] both in aggregate and individual level. If several elicitation experimental data are poor-quality, then they will be abandoned by using some judgment methods. The detailed context is presented in the individual data of Section 4.

Based on the above assumption, a PWF model is built as follows:

$$N(p) = a_1 + \sum_{i=2}^{n} \left(a_i \prod_{j=1}^{i-1} (p - p_j) \right), \tag{32}$$

where the parameters $a_i \, (i = 1, 2, \cdots, n)$ are determined by Eq. (28) based on the obtained high-quality EPPs (i.e., Eq. (31)). The following expressions are presented:

$$a_1 = w[p_1] = \widehat{w}(p_1),$$

$$a_i = w[p_1, p_2, \cdots, p_i] = \sum_{r=1}^{i} \left(\prod_{\substack{s=1 \\ s \neq r}}^{i} \frac{\widehat{w}(p_r)}{(p_r - p_s)} \right) \quad i = 2, \cdots, n.$$

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[3] Newton interpolation is a numerical interpolation method that appears to use a rope across all fixed points (EPPs) to shape a curve. Therefore, the shaped curve is unstable because of the number of interpolation node. A detailed description is provided in Section 2.3.

[4] C-E method, which is defined in Definition 3 of Section 3, is based on the indifference of preference relation, which defines the utility of decision-maker, refer to the Section 3.1 for more details. We only provide the example such as prospect (0, 0.95; $100, 0.05) ∼ ($14, 1), and probability p = 0.05 corresponds to probability weight $\widehat{w}(p) = 14/100 = 0.14$.

[5] The experiment or field questionnaire survey was carried out to obtain detailed information on the value and weighting functions of Tversky and Kahneman (1992), Gonzalez and Wu (1999), Abdellaoui et al. (2008), and so on.

[6] High-quality experimental data indicate that these data are obtained carrying out a good design experiment, and these data could respond to the preferences of the decision-maker in most occasions.

Integrating Eq. (33) to Eq. (32), a numerical PWF model is presented as follows:

$$N(p) = \widehat{w}(p_1) + \sum_{i=2}^{n} \left(\sum_{r=1}^{i} \prod_{\substack{s=1 \\ s \neq r}}^{i} \left(\frac{\widehat{w}(p_r)}{p_r - p_s} \right) \prod_{j=1}^{i-1} (p - p_j) \right). \tag{33}$$

In Eq. (33), $p_1, p_2, \cdots, p_n$ are the probabilities of event outcomes $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$. Eq. (33) is a known function without making any parametric assumptions as given in n EPPs. The advantage of this method is that the PWF is determined by EPPs, and its shape varies with EPPs. Moreover, it provides a direct link between preferences and utilities.

In reality, the different situations lead to different choices of the decision makers, which imply that the form of PWF varies with the choices of decision maker. The expressions of parameter PWF are fixed when its parameters are determined. As a result, the built model is better to reflect and predict the risk preferences of decision-maker. To circumvent this problem, the proposed Newton interpolation is used to approximate PWF when the elicitation experimental data are increasing or changing. For instance, it assumes that m new EPPs are collected under the condition of known Eq. (31):

$$\left(p_{n+1}, \widehat{w}(p_{n+1}) \right), \left(p_{n+2}, \widehat{w}(p_{n+2}) \right), \cdots, \left(p_{n+m}, \widehat{w}(p_{n+m}) \right).$$

For convenience, the new increasing m EPPs are assumed to be different from the previous n EPPs. In other words, if $i \neq j$, then probabilities $p_i \neq p_j (i, j \in 1, 2, \cdots, n + m)$. m new coefficients could be solved by using Eq. (36). The new model based on Newton interpolation is expressed as follows:

$$\tilde{N}(p) = a_1 + \sum_{i=2}^{n} \left(a_i \prod_{j=1}^{i-1} (p - p_j) \right) + \sum_{i=n+1}^{n+m} \left(a_i \prod_{j=1}^{i-1} (p - p_j) \right) = N(p) + E(p), \tag{34}$$

where $N(p)$ is expressed in Eq. (33), $E(p)$ is an increasing term, and its form is shown as follows:

$$E(p) = \sum_{i=n+1}^{n+m} \left(\sum_{r=1}^{i} \prod_{\substack{s=1 \\ s \neq r}}^{i} \left(\frac{\widehat{w}(p_r)}{p_r - p_s} \right) \prod_{j=1}^{i-1} (p - p_j) \right).$$

Through Eq. (34), it is easy to find that the numerical model $\tilde{N}(p)$ is composed of two parts $N(p)$ and $E(p)$. Increase new EPPs could be considered as the modification of the previous numerical PWF model $N(p)$. Therefore, to compute for the increasing term, $E(p)$ is required. This method is not only convenient for our computation, but is also a response to the change in the risk preferences of decision maker with the change in EPPs $\left(p_i, \widehat{w}(p_i) \right)$ $(i = 1, 2, \cdots, n)$ and their number. Compared with parameter PWF models, the numerical PWF model does not have a specific parametric form. Its form varies with the elicitation EPPs to response to the risk preferences of decision-maker.

It is worth noting that using Newton interpolation to approximate PWF, the obtained expressions such as Eqs (32) and (33) are nonparametric and numerical models. However, we do not call this method nonparametric numerical approach because we only use Newton in-

terpolation method to interpolate the elicitation EPPs. The obtained model could have some problems at sometimes because of the Runge phenomenon[7] of high-order Newton interpolation polynomial. For instance, the built model could be unstable, the obtained risk preference curve may not be an inverse S-shape, or the properties of the built model could not satisfy the properties such as overweighting for small probabilities, underweighting for moderate and large probabilities defined in Definition 2. Moreover, with the use of Newton interpolation to interpolate a small number of elicitation EPPs, the approximated PWF model could not be accurately response to the risk preferences of decision-maker (it has been described in Subsection 3.1). Therefore, Subsection 3.4 proposes a new algorithm to increase several new PPs (i.e., CPPs) to solve the above problems and develop a good performance model to reflect and predict the risk preferences of the decision-maker. To distinguish the new model from the previous model $N(p)$, it denotes $IN(p)$ as the new nonparametric numerical model, the used approach is called as nonparametric numerical approach.

3.4. Nonparametric numerical approach

In actual decisions, the number of the elicitation EPPs directly affects the performance of the built model, which is approximated by curve fitting the elicitation EPPs or other methods such as nonparametric elicitation. In general, as more high-quality EPPs are collected, an improved PWF model will be developed to respond accurately to the risk preferences of the decision-maker. However, under the condition of facing same decision-making environment, the decision-maker may not provide extensive preference information under most of the time in real life. If the number of the elicitation EPPs collected by elicitation method (C-E method) is small, then Newton interpolation approach is used to interpolate these elicitation EPPs directly, the developed numerical PWF model cannot approximate PWF with good performances at the shape and properties.

In this section, a nonparametric numerical approach is proposed to approximate PWF. The essence of this approach is the use of Newton interpolation to approximate PWF based on two kinds of different PPs. According to the difference of the number of collected elicitation EPPs, this approach consists of two parts. In the first part, a small number of elicitation EPPs are considered to approximate PWF model. A new algorithm is utilized to add several CPPs, and then Newton interpolation is used to interpolate these EPPs and CPPs. Part 2 takes account of a large number of elicitation EPPs, it proposes a piecewise Newton interpolation approach to interpolation the obtained elicitation EPPs, such that a piecewise nonparametric numerical PWF model is built.

In the process of increasing CPPs, the proposed new algorithm combines the existing parameter PWF models (for convenience, several parameter models are shown in Table 1 under the theoretic framework of cumulative prospect theory) and the elicitation EPPs to

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[7] In the early twentieth century, Runge discovered the Runge phenomenon (Carnahan et al., 1969): along with the increase in the interpolation node number. The interpolation function and fitted curve are not close, and the interpolation function is unstable based on the oscillation behavior of the high-order Newton interpolation polynomial.

infer CPPs. Thus, the collected CPPs have the characteristics of the parameter model, and are related to the elicitation EPPs. Based on these reasons, using Newton interpolation to fit the elicitation EPPs and the inferred CPPs, the developed model not only accurately reflects the elicitation EPPs, but only has the advantages of parameter model. Intuitively, Figure 5 shows the flowchart of the use of nonparametric numerical approach to approximate PWF.
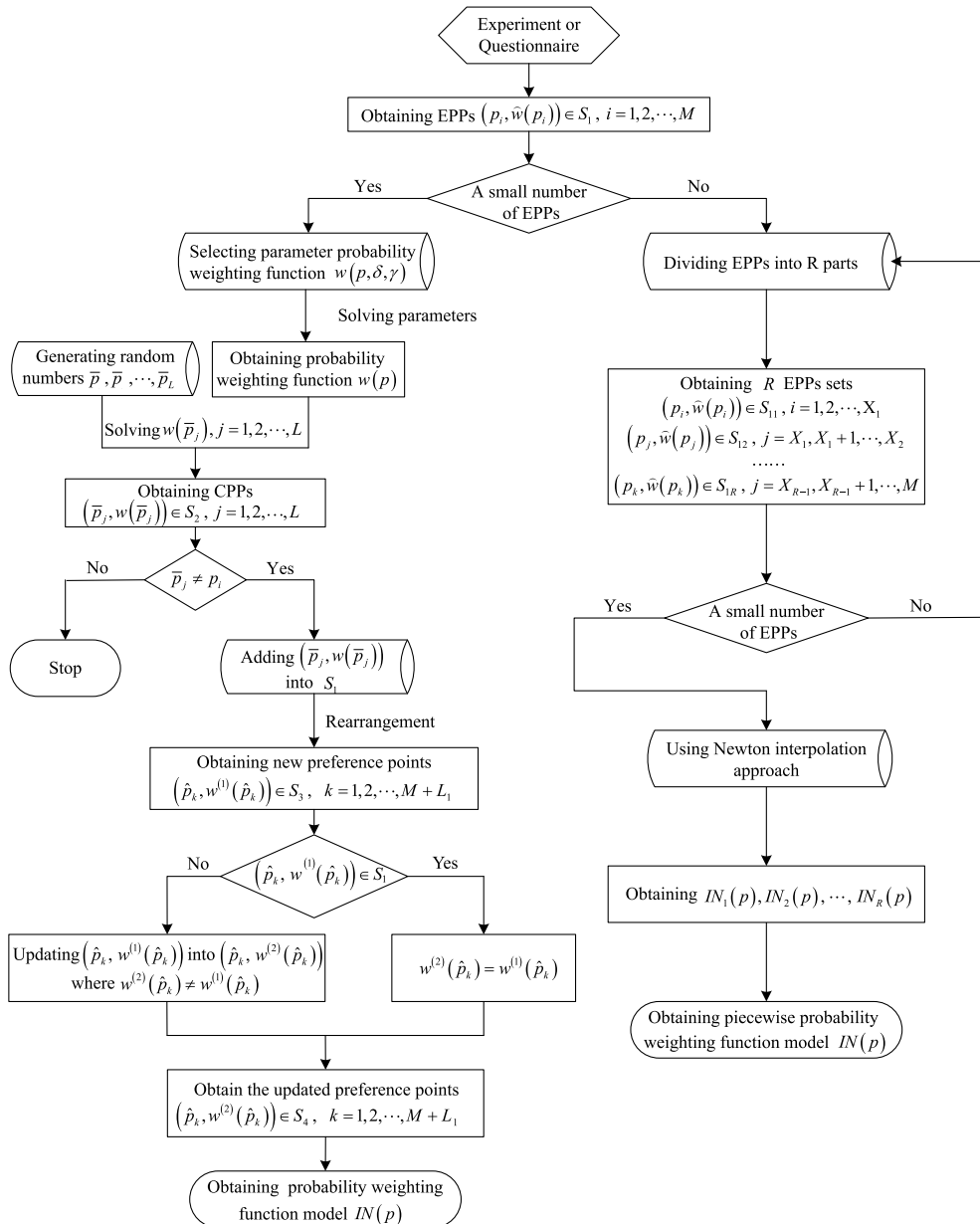

Figure 5. Building model flow chart

Table 1. Several parametric specifications of PWF

Parameter	Function Form	Parameter Information
One-parameter model	$w_+(p) = \dfrac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}$, $w_-(p) = \dfrac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}$	Tversky and Kahneman (1992): Given a power function with two parameters γ and δ, and $0 < \gamma, \delta < 1$. To fix them, at last two PPs should be selected.
	$w_+(p) = \exp[-(-\ln p)^\gamma]$, $w_-(p) = \exp[-(-\ln p)^\delta]$	Prelec (1998): Given an exponential function with two parameters α and β, and $0 < \alpha, \beta < 1$.
Two-parameter model	$w_+(p) = \dfrac{\delta^+ p^{\gamma^+}}{\delta^+ p^{\gamma^+} + (1-p)^{\gamma^+}}$, $w_-(p) = \dfrac{\delta^- p^{\gamma^-}}{\delta^- p^{\gamma^-} + (1-p)^{\gamma^-}}$	Tversky and Fox (1995): Given a power function with four parameters $0 < \gamma^+, \gamma^- < 1$, and $\delta^+, \delta^- > 0$. To fix them, at last four PPs should be selected.
	$w_+(p) = \exp[-\delta^+(-\ln p)^{\gamma^+}]$, $w_-(p) = \exp[-\delta^-(-\ln p)^{\gamma^-}]$	Prelec (1998): Given an exponential function with four parameters $0 < \gamma^+, \gamma^- < 1$, and $\delta^+, \delta^- > 0$. To fix them, at last four PPs should be selected.
Three-parameter model	$w_+(p) = \begin{cases} c^+ p^{\delta^+}, & \text{if } p \le \eta^+ \\ 1 - d^+ (1-p)^{\gamma^+}, & \text{if } p > \eta^+ \end{cases}$, $w_-(p) = \begin{cases} c^+ p^{\delta^-}, & \text{if } p \le \eta^- \\ 1 - d^+ (1-p)^{\gamma^-}, & \text{if } p > \eta^- \end{cases}$	Diecidue et al. (2009): For $c = \eta^{-\delta} \gamma \eta / (\gamma \eta + \delta(1-\eta))$, and $d = (1-\eta)^{-\gamma} \delta(1-\eta) / (\gamma \eta + \delta(1-\eta))$, six parameters exist $\delta^+, \delta^-, \gamma^+, \gamma^- > 0$, and $0 < \eta^+, \eta^- < 1$. To fix them, at last six PPs should be selected.

Part 1: A small number of EPPs

Parameter models were described in Figures 2–4. For convenience, these models are described again in Table 1 with gains and losses. Furthermore, Table 1 shows that the parameters of the proposed parameter PWF models would be determined by a small number of EPPs. We propose the use of existing parameter models in Table 1 and the obtained EPPs in Eq. (36) to add several new CPPs into the EPPs.

The following process is used to determine the risk preferences of decision maker on the basis of limited EPPs, the processes are shown as follows: At first, few limited elicitation EPPs are used to determine the parameters of $w(p)$ defined in Table 1 with the aid of mathematical software, such as MATLAB. At second, several CPPs are collected by using an algorithm after the parameters of model $w(p)$ are fixed. At third, several CPPs are added into the set of EPPs. At fourth, the added CPPs are updated on the basis of the collected elicitation EPPs in order to obtain several high-quality CPPs. At last, the collected PPs include elicitation EPPs and CPPs. Thus, a new nonparametric numerical PWF model could be developed by using Newton interpolation approach to interpolate these PPs.

The three main building blocks of this method are obtaining elicitation EPPs, generating and updating CPPs and using Newton interpolation method to approximate PWF. The first

main building block is important in eliciting the function to collect high-quality experimental data, which determine the performance of the developed model. Researchers can gather several experimental data through eliciting experiment or field questionnaire survey. The number of the collected elicitation EPPs is assumed to be small[8]. For convenience, the number of EPPs $n \leq 11$ is small in terms of preference information (of course, n could have a larger positive integer when the researcher asks a good performance to reflect and predict the risk preferences of decision-maker). A nonparametric numerical approach is used to develop the model. The procedure for explaining the nonparametric numerical model consists of nine steps and is summarized in the following section.

The nonparametric numerical approach is processed as follows:

Step 1. An experiment or a questionnaire survey is elicited. A set of experimental data could be collected from an experiment or a questionnaire survey (see Table 3 in Tversky and Kahneman (1992) for reference). For convenience, it assumes that M pairs experimental data have been collected.

Step 2. PWF is elicited to obtain EPPs defined in Definition 4. The C-E method introduced in Definition 3 of Section 3 is used to transform monetary outcomes into psychology probability $\widehat{w}(p_i)$ $(i = 1, 2, \cdots, M)$ with probability p_i $(i = 1, 2, \cdots, M)$. The obtained elicitation EPPs are represented by dyadic arrays $\left(p_i, \widehat{w}(p_i)\right)$ $(i = 1, 2, \cdots, M)$, the set of elicitation EPPs is denoted by S_1, that is, $\left(p_i, \widehat{w}(p_i)\right) \in S_1 (i = 1, 2, \cdots, M)$.

Step 3. A parameter model is selected. Three parameter PWF models are presented in Table 1. We only select one of them according to the different number of parameters[9]. For convenience, it assumes that the selected parameter model is a two-parameter[10] PWF model $w(p, \delta, \gamma)$.

Step 4. The parameters δ and γ are solved. The least square method or curve fitting method may be used to solve the parameters δ and γ by means of the collected elicitation EPPs. For convenience, the determined parameter model is denoted by $w(p)$.

Step 5. Random numbers are generated. To predict the preference information, a collection of random numbers $\overline{p}_j$ $(j = 1, 2, \cdots, L; \overline{p}_1 = 0, \overline{p}_L = 1)$ is generated on interval $\begin{bmatrix} 0, & 1 \end{bmatrix}$ with the aid of MATLAB. For convenience, the generated random numbers can be in either a uniform or piecewise uniform distribution according to the density degree of

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[8] The large or small number of collected EPPs depends on the distribution density of probability p on interval [0, 1] and the need of decision-maker for the accuracy.

[9] When decision-maker faces positive and negative prospects, the positive prospect is deems as gains, the negative prospect is deemed as losses. Decision-maker exhibits different psychological reflections for different prospects. Thus, we need to solve the parameters of different probability weighting function models $w_+(p)$ for positive prospect and $w_-(p)$ for negative prospect. The existing parameter probability weighting function models such as Eqs (20)–(24), which correspond to the expressions of positive and negative prospects, are shown in Table 1.

[10] When the selected parameter probability weighting function model is the one-parameter model, it is denoted by $w(p, \delta)$; when the selected parameter model is three-parameter model, it is denoted by $w(p, \delta, \gamma, \eta)$. The two-parameter model is selected in this paper because of a good performance, that is, two parameters controlling risk attitude's two aspects. The model was introduced in detailed in literature review.

$\overline{p}_j\left(j=1,2,\cdots,L\right)$ [11], but the generated random numbers must satisfy the following property: for all $i=1,2,\cdots,M$, $\overline{p}_j = p_i\left(j=1,2,\cdots,L\right)$.

Step 6. Several objective probabilities are added. If probability $\overline{p}_j \neq p_i\left(j=1,2,\cdots,L; i=1,2,\cdots,M\right)$, then it adds the probability $\overline{p}_j$ into the set of p_i, it assumes that the number of $\overline{p}_j \neq p_i$ is $L_1\left(L_1 \leq L\right)$, so the number of obtained probabilities is $M+L_1\left(L_1 \leq L\right)$. For convenience, it denotes that the sequence of probability is $\hat{p}_k\left(k=1,2,\cdots,M+L_1\right)$.

Step 7. Several CPPs are solved. The probabilities $\hat{p}_k\left(k=1,2,\cdots,M+L_1\right)$ obtained in *Step 6* and the parameter model $w(p)$ determined in **step 4** are used to compute $w\left(\hat{p}_k\right)\left(k=1,2,\cdots,M+L_1\right)$, thus a series of PPs $\left(\hat{p}_k, w\left(\hat{p}_k\right)\right)\left(k=1,2,\cdots,M+L_1\right)$ can be collected. Specially, the collection of PPs is denoted by the set S_2, that is, $\left(\hat{p}_k, w\left(\hat{p}_k\right)\right) \in S_2$ $\left(k=1,2,\cdots,M+L_1\right)$. For convenience, we call the collection of PPs as CPPs.

Step 8. Update CPPs. To measure the risk preferences of decision-maker, the obtained CPPs should be updated according to the elicitation EPPs. Detailed preference information updating process is organized as follows.

At first, some CCPs are replaced with EPPs by using the following equation:

$$w^{(1)}\left(\hat{p}_k\right) = \begin{cases} \widehat{w}\left(p_i\right), & \hat{p}_j = p_i \\ w\left(\hat{p}_j\right), & \hat{p}_j \neq p_i \end{cases}, \tag{35}$$

where $i=1,2,\cdots,M$; $j=1,2,\cdots,M+L$. The obtained PPs are denoted by $\left(\hat{p}_k, w^{(1)}\left(\hat{p}_k\right)\right)$ $\left(k=1,2,\cdots,M+L_1\right)$, and the set is denoted by S_3, which means that the PPs $\left(\hat{p}_k, w^{(1)}\left(\hat{p}_k\right)\right) \in S_3$ $\left(k=1,2,\cdots,M+L_1\right)$.

At second, the values of some CPPs are updated. Given $H\left(H \leq L_1\right)$ PPs that have been interpolated between the two adjacent EPPs $\left(\hat{p}_k, w^{(1)}\left(\hat{p}_k\right)\right)$ and $\left(\hat{p}_{k+H+1}, w^{(1)}\left(\hat{p}_{k+H+1}\right)\right)$, that is, $\left(\hat{p}_k, w^{(1)}\left(\hat{p}_k\right)\right)$, $\left(\hat{p}_{k+H+1}, w^{(1)}\left(\hat{p}_{k+H+1}\right)\right) \in S_1, S_3$ are EPPs; $\left(\hat{p}_{k+1}, w^{(1)}\left(\hat{p}_{k+1}\right)\right)$, $\cdots$, $\left(\hat{p}_{k+H}, w^{(1)}\left(\hat{p}_{k+H}\right)\right) \in S_2, S_3$ are CPPs). To well reflect and predict the risk preferences of decision-maker, the CPPs $\left(\hat{p}_{k+h}, w^{(1)}\left(\hat{p}_{k+h}\right)\right)$ $\left(k=1,2,\cdots,M+L_1-h; h=1,2,\cdots,H\right)$ should be modified according to the collected EPPs $\left(\hat{p}_k, w^{(1)}\left(\hat{p}_k\right)\right)$ and $\left(\hat{p}_{k+H+1}, w^{(1)}\left(\hat{p}_{k+H+1}\right)\right)$. For convenience, the updated PPs are denoted by $\left(\hat{p}_k, w^{(2)}\left(\hat{p}_k\right)\right)\left(k=1,2,\cdots,M+L_1\right)$, where,

$$w^{(2)}\left(\hat{p}_k\right) = w^{(1)}\left(\hat{p}_k\right), \quad k=1,2,\cdots,M+L_1-H-1, \tag{36}$$

$$w^{(2)}\left(\hat{p}_{k+1}\right) = w^{(1)}\left(\hat{p}_k\right) + f\left(w\left(\hat{p}_k\right)\right),$$

$$\cdots\cdots$$

$$w^{(2)}\left(\hat{p}_{k+h}\right) = w^{(1)}\left(\hat{p}_{k+h-1}\right) + f\left(w\left(\hat{p}_{k+h-1}\right)\right), \quad h=1,2\cdots,H,$$

$$\cdots\cdots$$

$$w^{(2)}\left(\hat{p}_{k+H+1}\right) = w^{(1)}\left(\hat{p}_{k+H}\right) + f\left(w\left(\hat{p}_{k+H+1}\right)\right) = w^{(1)}\left(p_{k+H+1}\right).$$

---

[11] Probability weighting function reflects decision makers' preference under risk, the change of preference is relatively fast at the endpoint 0 and 1. To adapt this change, piecewise uniform distribution's probabilities are selected. If the selected probability $\overline{p}$ is more intensive, uniform distribution's probabilities should be selected.

According to the inverse-S property of the PWF (Wu & Gonzalez, 1996, 1999), the function *f* can be built as follows:

$$f\left(w^{(1)}\left(\hat{p}_{k+h}\right)\right) = T \cdot \frac{\Delta x_k}{w\left(\hat{p}_{k+H+1}\right) - w\left(\hat{p}_k\right)}, \tag{37}$$

$$\Delta x_k = w^{(1)}\left(\hat{p}_{k+h+1}\right) - w^{(1)}\left(\hat{p}_{k+h}\right) = w\left(\hat{p}_{k+h+1}\right) - w\left(\hat{p}_{k+h}\right),$$

$$T = w^{(1)}\left(\hat{p}_{k+H+1}\right) - w^{(1)}\left(\hat{p}_k\right),$$

where $h = 1, 2, \cdots, H$.

The updated sequence PPs $\left(\hat{p}_k, w^{(2)}\left(\hat{p}_k\right)\right)\left(k = 1, 2, \cdots, M + L_1\right)$, and their set is denoted by S_4.

$$\left(\hat{p}_k, w^{(2)}\left(\hat{p}_k\right)\right) \in S_4, \quad k = 1, 2, \cdots, M + L_1. \tag{38}$$

Step 9. Build nonparametric numerical PWF model. On the basis of the obtained PPs and Eq. (27), the coefficients for difference quotients can be solved.

$$a_1 = w\left[\hat{p}_1\right] = w^{(2)}\left(\hat{p}_1\right),$$

$$a_k = w\left[\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_{k+1}\right] = \sum_{i=1}^{k}\left(\prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{w^{(2)}\left(\hat{p}_i\right)}{\left(\hat{p}_i - \hat{p}_j\right)}\right), \quad k = 2, 3, \cdots, M + L_1. \tag{39}$$

Taking the above coefficients into Eq. (28), the approximated nonparametric numerical PWF model is exhibited as follows:

$$IN(p) = w^{(2)}\left(\hat{p}_1\right) + \sum_{i=2}^{M}\left(\sum_{r=1}^{i}\prod_{\substack{s=1 \\ s \neq r}}^{i}\left(\frac{w^{(2)}\left(\hat{p}_r\right)}{\hat{p}_r - \hat{p}_s}\right)\prod_{j=1}^{i}\left(p - \hat{p}_j\right)\right) +$$

$$\sum_{i=M+1}^{M+L_1}\left(\sum_{r=1}^{i}\prod_{\substack{s=1 \\ s \neq r}}^{i}\left(\frac{w^{(2)}\left(\hat{p}_r\right)}{\hat{p}_r - \hat{p}_s}\right)\prod_{j=1}^{i}\left(p - \hat{p}_j\right)\right). \tag{40}$$

On the basis of solved Eq. (40), if researcher obtains *m* groups new elicitation EPPs which are not in Set S_1, to update the nonparametric numerical model, the method which is same to Eq. (34) is used, a new nonparametric numerical PWF model is built as follows:

$$IN^{(1)}(p) = IN(p) + IE(p). \tag{41}$$

In where, *IN(p)* is the expression of Eq. (40), *IE(p)* is the adding term and is shown as follows:

$$IE(p) = \sum_{i=M+L_1+1}^{M+L_1+m}\left(\sum_{r=1}^{i}\prod_{\substack{s=1 \\ s \neq r}}^{i}\left(\frac{w^{(2)}\left(\hat{p}_r\right)}{\hat{p}_r - \hat{p}_s}\right)\prod_{j=1}^{i}\left(p - \hat{p}_j\right)\right). \tag{42}$$

In general, the models of Eqs (41) and (42) improved the models of Eqs (33) and (34), respectively. The difference between the nonparametric numerical approach in Section 3.4

and the numerical method in Section 3.3 is that the built model increases several CPPs and relates these CPPs to the elicitation EPPs. This approach makes a contribution that the approximation PWF could better reflect decision-maker's risk preferences and thus better predict decision-maker's behaviors.

Part 2: A large numbers of EPPs

In the case of known large numbers of the elicitation EPPs, the proposed nonparametric numerical approach to approximate the PWF shows good performance in response to the risk preferences of decision-maker. The proposed approach could accommodate a change in decision-maker's risk preferences both for aggregate data and for individual data (the empirical analysis of Section 4 demonstrates this conclusion). However, extensive information on preferences is generally difficult to gather, Gonzalez (1993) uses a large amount of preferences information to fit the parameter PWF.

Assume that large numbers of EPPs have obtained. The steps in using nonparametric numerical approach to approximate PWF are present as follows:

Step I. Elicit PWF. The step is similar to **Step 1** in that it assumes that the obtained elicitation EPPs are $\left(p_i\, \widehat{w}\left(p_i\right)\right)$, their number is $M\left(M>10\right)$ and their set is denoted by S_1.

$$\left(p_i,\widehat{w}\left(p_i\right)\right)\in S_1,\quad j=1,2,\cdots,M. \tag{43}$$

Step II. Divide the preference information into $R\left(R\geq2\right)$ parts. According to the size of probability $p_i\left(i=1,2,\cdots,M;p_i\in\left(0,1\right)\right)$, the EPPs are divided into $R\left(R\geq2\right)$ parts. The detailed context is given as follows:

$$
\begin{aligned}
&\left(p_i,\widehat{w}\left(p_i\right)\right)\in S_{11},\quad i=1,2,\cdots,X_1,\\
&\left(p_j,\widehat{w}\left(p_j\right)\right)\in S_{12},\quad j=X_1,X_1+1,\cdots,X_2,\\
&\cdots\cdots\\
&\left(p_k,\widehat{w}\left(p_k\right)\right)\in S_{1R},\quad k=X_{R-1},X_{R-1}+1,\cdots,M.
\end{aligned} \tag{44}
$$

Assuming $R=3$, the preference information is divided into three parts because of the different reaction of PWF for small probability, intermediate probability and big probability. The divided intervals for probability p are $p_i\in(0,0.1]$, $p_j\in[0.1,0.9]$ and $p_k\in[0.9,1)$ respectively. The corresponding preference information $\left(p,\widehat{w}\left(p\right)\right)$ is also divided into three parts, namely, S_{11}, S_{12} and S_{13}. To ensure the continuity of PWF, we take the boundary points on every intervals such that the total number of preference information is increased to $M+2$ because the EPPs $\left(0.1,\widehat{w}\left(0.1\right)\right)$ and $\left(0.9,\widehat{w}\left(0.9\right)\right)$ are used twice.

$$
\begin{aligned}
&\left(p_i,\widehat{w}\left(p_i\right)\right)\in S_{11},\quad i=1,2,\cdots,X_1,\\
&\left(p_j,\widehat{w}\left(p_j\right)\right)\in S_{12},\quad j=X_1,X_1+1,\cdots,X_2,\\
&\left(p_k,w\left(p_k\right)\right)\in S_{13},\quad k=X_2,X_2+1,\cdots,M.
\end{aligned}
$$

Step III. Build the piecewise PWF model. We build our nonparametric numerical model in each interval according to the preference information of three intervals.

$$IN_1(p) = \widehat{w}(p_1) + \sum_{i=2}^{X_1}\left(\sum_{r=1}^{i}\prod_{\substack{s=1\\s\neq r}}^{i}\left(\frac{\widehat{w}(p_r)}{p_r - p_s}\right)\prod_{t=1}^{i}(p - p_t)\right),$$

$$IN_2(p) = \widehat{w}(p_{X_1}) + \sum_{j=X_1+1}^{X_2}\left(\sum_{r=X_1}^{j}\prod_{\substack{s=X_1\\s\neq r}}^{j}\left(\frac{\widehat{w}(p_r)}{p_r - p_s}\right)\prod_{t=X_1}^{j}(p - p_t)\right), \qquad (45)$$

......

$$IN_R(p) = \widehat{w}(p_{X_R}) + \sum_{k=X_R+1}^{M}\left(\sum_{r=X_R}^{k}\prod_{\substack{s=X_R\\s\neq r}}^{k}\left(\frac{\widehat{w}(p_r)}{p_r - p_s}\right)\prod_{t=X_R}^{k}(p - p_t)\right),$$

where the middle EPPs $\left(p_{X_1},\widehat{w}(p_{X_1})\right),\left(p_{X_2},\widehat{w}(p_{X_2})\right),\cdots,\left(p_{X_{R-1}},\widehat{w}(p_{X_{R-1}})\right)$ are used twice. This method ensures that the piecewise PWF curve $IN_r(p)$ intersects the curve $IN_{r+1}(p)$ at $\left(p_{X_r},\widehat{w}(p_{X_r})\right)(r=1,2\cdots,R-1)$. Thus, the model with R parts ensures the continuity of elicitation PWF in interval (0, 1). Moreover, the piecewise nonparametric numerical PWF model is built as follows.

$$IN(p) = \begin{cases} IN_1(p), & p \in \left(0\,,\, p_{X_1}\right] \\ IN_2(p), & p \in \left[p_{X_1}, p_{X_2}\right] \\ \quad\cdots\cdots \\ IN_R(p), & p \in \left[p_{X_{R-1}}, 1\right) \end{cases}. \qquad (46)$$

In this part, the purpose of using piecewise Newton interpolation method is to avoid the oscillation behavior of high-order Newton polynomial. We can also increase several CPPs by using **Steps 1–6**, then use **Step II–III** to obtain the piecewise model. The framework to approximate PWF with the aid of the nonparametric numerical approach under the condition of known large numbers of the elicitation EPPs is shown in Figure 5.

In summary, the nonparametric numerical approach could better approximate PWF to ensure good performance of the model to reflect and predict the risk preferences of decision-maker. This approach approximates PWF by interpolating the elicitation EPPs and the inferred CPPs without assuming a specific parametric form. Therefore, the developed PWF model varies with the risk preferences of decision-maker. In the process, obtaining the elicitation EPPs by using C-E method is important. Thus, decision making requires an experiment or questionnaire survey before decision-making theories are used with the proposed nonparametric numerical approach. Although this approach could increase the cost of decision-making, it is good at reflecting and predicting the risk preferences of decision-maker so that researcher predicts decision-maker's behaviors and provides a good decision-making scheme. This is also the goal of our study.

4. Empirical analysis

Empirical analysis indicates that the proposed nonparametric numerical approach has a good performance at approximating PWF, and that the built model can better reflect and predict decision-maker's risk preferences by satisfying the EPPs and basic properties of PWF both for aggregate data and for individual data. As a result of the limited condition, empirical analysis used the experiment data from Tversky and Kahneman 1992 for the aggregate subject data and Wu and Gonzalez (1999) for the individual subject data. A comparative analysis of Prelec's two-parameter model, numerical model (using Newton interpolation method to interpolate the elicitation EPPs) and nonparametric numerical model indicates that the proposed nonparametric numerical PWF model performs well both in aggregate and individual subject level.

Data processing and updating were conducted to adapt to the change in experimental data (For more details, one may refer to **Parts 1** and **2**). This method of data processing and data updating is not only convenient our solving, but also dynamic response to the preferences of decision-maker. As long as the decision-maker offers several choices under the condition of eliciting experiment or questionnaire survey, the elicitation EPPs could be collected, and a nonparametric numerical approach could be used to approximate PWF and build a nonparametric numerical model. Based on the built nonparametric numerical model, researchers could know the risk preferences of decision-maker during the actual decision making analysis, such that a decision-making scheme could be made to sever economic decision-making. This condition is also an important advantage for the proposed nonparametric numerical approach because examining the observed choices provides insight into the psychological reasoning that underlies the survey data both at the aggregate and individual level (the analysis will be shown in the following Subsections).

4.1. Aggregate data

Considering most people's decision-making, it requires large experiment data that reflect their risk preferences to verify the proposed nonparametric numerical approach. In this subsection, we quote the high-quality data from Tversky and Kahneman (1992)[12], which benefited from interviews with 25 graduate students from Berkeley and Stanford who have no special training in decision theory. The respondents were asked to imagine that they are actually faced with the choice described in the problem, and to indicate the decision they would have made in such a case. These experimental data are displayed in Table 1. Although these experimental data were collected a long time ago (in 1992), they can still well reflect most people's decision-making and can be used to verify the validity of the proposed nonparametric numerical approach.

A C-E method was introduced in Definition 3 for translating monetary outcomes ε into the corresponding psychological probability $\widehat{w}(p)$ with objective probability p for experimental data in Table 3. Implementing the transformation of Eq. (30), the relationship of

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[12] In this experiment, each subject participated in three separate one-hour sessions that were several days apart, and each subject was paid $25 for participation.

probability p_i and the corresponding psychological probability $\widehat{w}(p_i)$ about event outcome ε_i are shown in Table 3. To distinguish the different reactions of decision-maker for gains-losses and low-high monetary outcome, the prospects are divided into four sets: positive low monetary outcome prospect ($0 < \varepsilon_i < 200$), positive high monetary outcome prospect ($200 \leq \varepsilon_i \leq 400$), negative positive low monetary outcome prospect ($-200 < \varepsilon_i < 0$) and negative high monetary outcome prospect ($-400 \leq \varepsilon_i \leq -200$). Once the sets of prospects are established, a quantitative description of these prospects is prepared to analyze the shape and properties of the approximation PWF shown in Definition 2.

EPPs defined in Definition 4 $(p_i, \widehat{w}(p_i))$, $i = 1, 2, \cdots, M$ can be collected by the C-E method defined in Definition 3 to process the experimental data in Table 2. Table 3 shows the elicitation EPPs of most people decision-making for the prospects of different levels.

Table 2. Median cash equivalents for all nonmixed prospects (Tversky & Kahneman, 1992)

outcomes	Probability								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
(0,50) (0,−50)	4 −3		9 −8		21 −21		37 −39		45 −47
(0,100) (0,−100)		14 −8		28 −23.5	36 −42	56 −63		78 −84	
(0,200) (0,−200)	10 −3		20 −23		76 −89		131 −155		188 −190
(0,400) (0,−400)	12 −14								377 −380
(50,100) (−50,−100)			59 −59		71 −71		83 −85		
(50,150) (−50,−150)		64 −60		72.5 −71	86 −92	102 −113		128 −132	
(100,200) (−100,−200)		118 −112		130 −121	141 −142	162 −158		178 −179	

Note: The two outcomes of each prospect are given in the left-hand side of each row; the corresponding value of median cash equivalents are given in the right columns (i.e., ($9, 1.0) is equivalent to (0, 0.90; $50, 0.10), the value of $9 in the upper left corner.).

Table 3. Relationship of probability p and psychological probability probability $w(p)$

Outcomes	Probability p and probability weight $w(p)$								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
$0 < \varepsilon < 200$	0.08	0.14	0.18	0.28	0.42	0.56	0.74	0.78	0.90
$200 \leq \varepsilon \leq 400$	0.05		0.10		0.38		0.66		0.94
$-200 < \varepsilon < 0$	0.06	0.08	0.16	0.24	0.42	0.63	0.78	0.84	0.94
$-400 \leq \varepsilon \leq -200$	0.02		0.12		0.45		0.78		0.95

Applying Newton interpolation

Using these elicitation EPPs in Table 3 and Newton interpolation approach, a nonparametric PWF can be expressed as follows:

$$N(p) = w[p_1] + w[p_1, p_2, \cdots, p_i] \prod_{i=2}^{M} (p - p_{i-1}), \tag{47}$$

where $w[p_1], w[p_1, p_2], \cdots, w[p_1, p_2, \cdots, p_9]$ are difference quotients and are defined such as Eqs (28)–(30). From the above expressions (56), the developed PWF model is a nonlinear equation determined by the elicitation EPPs and varies with the number and value of the elicitation EPPs. For instance, the expressions of $N_1^+(p)$ and $N_1^-(p)$ are both eight-order nonlinear equations and are determined by nine elicitation EPPs, $N_2^+(p)$ and $N_2^-(p)$ are both four-order nonlinear equations and are determined by five elicitation EPPs. To intuitively show the shape and properties of the approximated PWF models, Figure 6 is given with the trend of $N_1^+(p)$, $N_2^+(p)$, $N_1^-(p)$ and $N_2^-(p)$ to reflect and predict the risk preferences of decision-maker.

In Figure 6, compared with $N_2^-(p)$, $N_1^-(p)$ exhibits more risk aversion; $N_1^+(p)$ is more risk-seeking than $N_2^+(p)$. At the same time, the four curves of Figure 6 exhibit overweighting for small probabilities and underweighting for moderate and big probabilities. The curves of $N_1^-(p)$ and $N_2^-(p)$ have a good performance at approximating the shape of PWF. However, the performances of $N_1^+(p)$ and $N_2^+(p)$ are poor. The result reveals the instability of Newton interpolation method based on the elicitation EEPs. Thus, a nonparametric numerical approach is proposed to approximate PWF to reflect and predict the risk preferences of decision-maker.
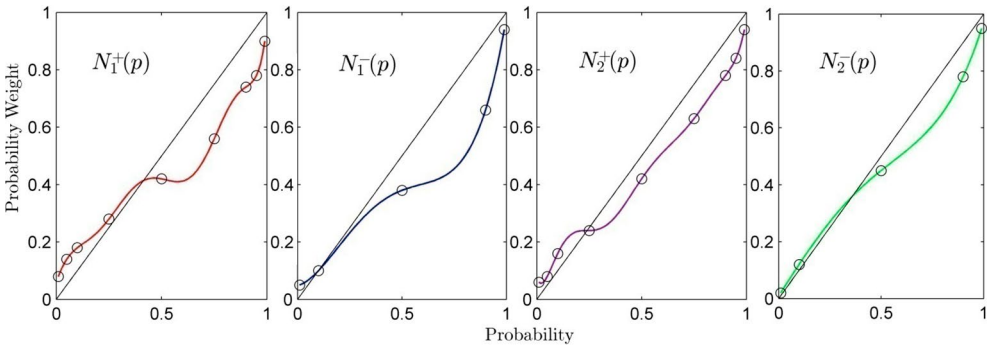

Figure 6. The approximation PWF model based on Newton interpolation method

Applying nonparametric numerical approach

According to the obtained elicitation EPPs in Table 3, the proposed nonparametric numerical approach was used to approximate PWF. It is obviously that the number of the elicitation EPPs is a small number, so the nonparametric numerical model could be approximated by using **Step 1–8.** A two-parameter PWF model is selected as follows

$$w(p, \delta, \gamma) = \exp\left[-\delta \left(-\ln p \right)^\gamma \right], \tag{48}$$

where, $w(p, \delta, \gamma)$ is a two-parameter Prelec's model (Prelec, 1998). Figure 7 exhibits the characteristic pattern of the three kinds of reflecting and predicting the risk preferences' PWF models.

First, the PWF curves indicate risk seeking for small probabilities (the PWF curves above diagonal) and risk aversion for intermediate and big probabilities (the curves below the diagonal)[13]. Thus, the PWF curves are referred to as inverse-S shape. Second, they show that, facing negative prospect (–200 < ε < 0), decision-maker give more weight to small probabilities and less weight to intermediate and large probabilities. Last, the curves also indicate that preference relations satisfy continuity and monotonicity, and that slight departures from monotonicity about $N(p)$ and $IN(p)$ in the interval $\begin{bmatrix} 0.15, & 0.25 \end{bmatrix}$ could reflect the preference fluctuations of decision-maker from risk-seeking to risk-aversion.

However, compared with the existing parameter model $w(p)$, the numerical models $N(p)$ and the nonparametric numerical model $IN(p)$ could better reflect and predict risk preferences of decision-maker with the curves of $N(p)$ and $IN(p)$ going through the elicitation EPPs, which are obtained by eliciting an experiment or questionnaire survey and using C-E method. Moreover, compared with $N(p)$, $IN(p)$ performs well at the monotonicity of preference relation, and it also avoids oscillation of high-order Newton polynomial. Thus, Figure 7 exhibits intuitively the advantages and disadvantages of the three PWF models.

To verify that the model $IN(p)$ exhibits a good performance for different prospects' monetary outcomes[14], different curve graphs based on different PWF models are presented in Figure 8.
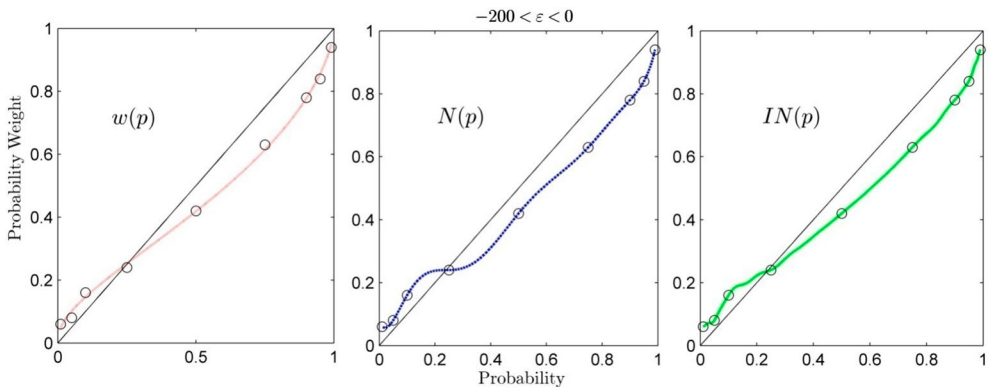

Figure 7. Curves of PWF based on $w(p)$, $N(p)$ and nonparametric numerical model $IN(p)$

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[13] It reflects the common features of people's attitude toward risk. People show different risk attitudes for different reference points. As we all know, EU theory's probability weighting function is $w(p) = p$, which exhibits a straight line (i.e., the diagonal). The risk attitude is defined by the spatial location relationships of preference curve $IN(p)$ and the diagonal $w(p) = p$: if the preference curve lies on the diagonal, decision-maker is risk-neutral; if the preference curve lies above the diagonal, decision-maker is risk-seeking; and if the preference curve lies below the diagonal, decision-maker is risk-aversion. For more details, one may refer to literature Tversky and Wakker (1995).

[14] The different prospects are mainly the comparison of gains the higher level monetary outcome (0 < ε < 200), gains the lower level monetary outcome 200 ≤ ε ≤ 400, losses the higher level monetary outcome –200 < ε < 0 and losses the lower level monetary outcome –400 ≤ ε ≤ –200.

Figure 8 outlines that the nonparametric numerical model $IN(p)$ that integrates the advantages of existing parameter PWF model $w(p)$ and the numerical PWF model $N(p)$ better reflect and predict decision-maker's risk preferences for every case.

Scholten and Read (2014) presented the "forgotten" fourfold pattern of risk preferences. Figure 9 demonstrates that the approximated PWF model $IN(p)$ satisfies the property of the Four-fold pattern of risk preference for decision-maker.
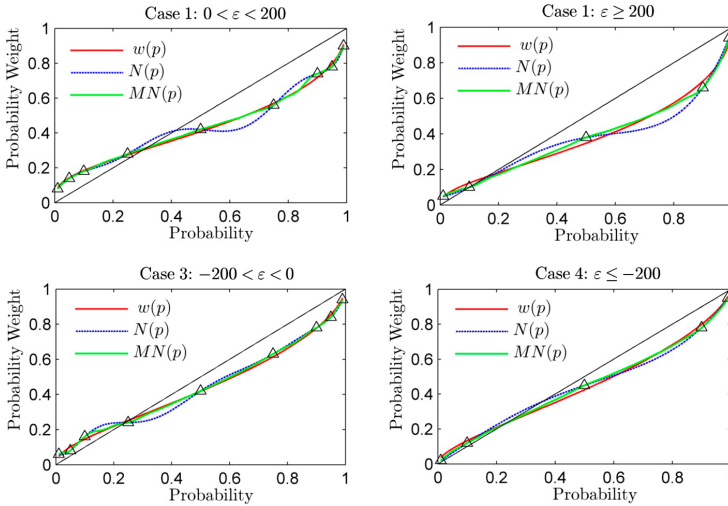

Figure 8. Curves of different models based on aggregate experimental data. $N_1^+(p)$ and $N_2^+(p)$, correspond to event outcomes of ε that lie below or above 200, respectively; both represent decision-maker's PWF for positive prospects, $N_1^-(p)$ and $N_2^-(p)$, correspond to event outcomes of ε that lie below or above –200, respectively; both show decision-maker's PWF for negative prospects
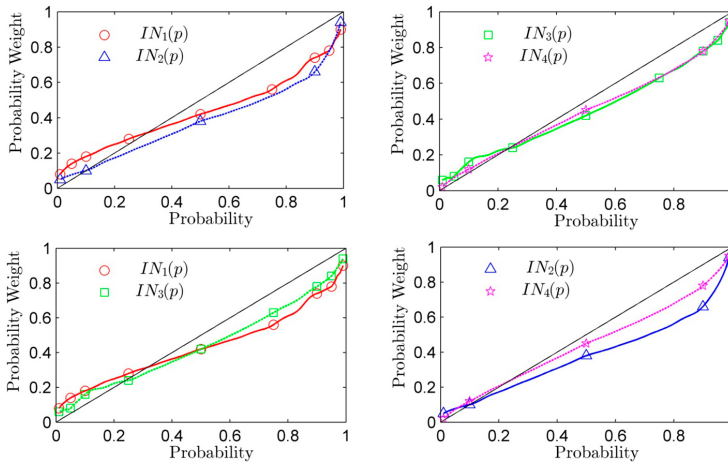

Figure 9. Curves of nonparametric numerical models based on aggregate experimental data. $IN_1(p)$ and $IN_2(p)$, respectively, correspond to event outcomes $0 < \varepsilon < 200$ and $\varepsilon \geq 200$, both represent decision-maker's PWF for positive prospects. $N_3(p)$ and $N_4(p)$, respectively, correspond to event outcomes $-200 < \varepsilon < 0$ and $\varepsilon < -200$, both show decision-maker's PWF for negative prospects

4.2. Individual data

The experiment data of Gonzalez and Wu (1999) are used to demonstrate that the proposed nonparametric numerical approach could be applied to approximate PWF in the individual subject level. The median certainty equivalent for each of the 165 two-outcome gambles appears in Table 4. The first column of Table 4 expresses two-outcome prospects; the first line expresses the probabilities of two-outcome prospects; others express the corresponding sure prospects under different probabilities.

To obtain the risk preference information of decision-maker, the C-E method defined in Definition 3 is applied to process these experimental data in Table 4. Figures 10–11 presents the parameter model $w(p)$ fitted by Eq. (48), numerical model $N(p)$ built by Eq. (33) and nonparametric numerical model $IN(p)$ approximated by using Eq. (48) for the median data shown in Table 4. The difference between Figures 10 and 11 is that Figure 11 abandons the non-high-quality elicitation EPP (0.6, 0.389) because it is far from the preference curve $w(p)$ in the first sub-graph of Figure 10. In this process, a two-parameter PWF $w(p)$ (it is same to the aggregate data) is selected as follows:

$$w(p) = \exp\left[\delta\left(-\ln p\right)^{\gamma} \right].\tag{49}$$

Parameters δ and γ are determined by using a least squares approach. The results include $\delta = 1.29$ and $\gamma = 0.32$ in Figure 10 and $\delta = 1.30$ and $\gamma = 0.32$ in Figure 11. It is obvious that $w(p)$ varies minimally with the number of EPPs, which means that the EPP (0.6, 0.389) is not high-quality experimental data. What is important is that it reveals the disadvantages of parameter PWF model, that is, assuming a specific parametric function form. $N(p)$ and $IN(p)$

Table 4. Median certainty equivalent for each monetary outcome from Gonzalez and Wu (1999)

Outcomes	Probability attached to higher outcome										
	0.01	0.05	0.10	0.25	0.40	0.50	0.60	0.75	0.90	0.95	0.99
25–0	**4.0**	4.0	8.0	9.0	10.0	9.5	12.0	11.5	14.5	13.0	19.0
50–0	**6.0**	7.0	8.0	12.5	10.0	14.0	12.5	19.5	22.5	27.5	40.0
75–0	**5.0**	10.0	8.5	14.0	17.0	16.0	18.0	23.0	36.0	31.5	48.5
100–0	**10.0**	10.0	15.0	21.0	19.0	23.0	35.0	31.0	63.0	58.0	84.5
150–0	**10.0**	10.0	18.0	25.0	34.0	25.0	49.0	70.0	41.5	106.0	118.0
200–0	**6.0**	9.0	21.0	26.5	34.5	34.0	56.0	48.0	80.0	102.0	158.0
400–0	**18.0**	24.0	24.0	54.0	64.0	58.0	58.0	115.0	208.0	249.0	277.0
800–0	**9.5**	42.0	36.0	90.0	91.0	89.5	207.0	197.5	404.0	448.5	519.0
50–25	**28.0**	29.5	30.5	32.0	34.5	34.5	37.5	36.5	38.0	41.5	41.0
75–50	**56.5**	58.0	56.0	59.5	62.0	63.0	64.0	64.5	64.5	65.0	68.5
100–50	**58.0**	59.0	59.0	62.5	66.5	70.0	78.0	82.5	80.0	78.5	89.0
150–50	**57.0**	58.5	71.0	79.0	89.0	84.0	92.0	99.0	121.0	106.0	117.0
150–100	**114.0**	110.0	116.5	116.0	121.0	125.0	125.5	131.0	133.5	130.0	142.5
200–100	**111.5**	115.0	117.0	123.0	131.0	135.0	144.0	146.0	149.0	171.0	158.0
200–150	**156.0**	165.0	160.0	166.0	171.0	170.5	176.0	177.0	187.5	179.5	190.0
$\widehat{w}(p)$	0.115	0.145	0.175	0.242	0.295	0.307	0.389	0.434	0.544	0.584	0.735

are both nonparametric numerical PWF model, their advantages and disadvantages have been expounded in aggregate data. Exhibiting the effect of high-quality experimental data is important in building a PWF model. A phenomenon that the curve of $N(p)$, but not $w(p)$ and $IN(p)$, changes significantly when abandoning non-high-quality experimental data could be discovered by comparing Figure 10 and 11. Moreover, $IN(p)$ shows good performance at the shape under the case of gaining high-quality elicitation EPPs than under the case of gaining non-high-quality elicitation EPPs. Therefore, several non-high-quality experimental data are abandoned based on whether they are far away from the fitting curve $w(p)$, or upon satisfaction of monotonicity of preference relation in the individual data.

Gonzalez and Wu (1999) collected a group of experimental data composed of 10 subjects with 165 two-outcome prospects per subject by using C-E method. Table 5 exhibits the processed experimental data of 10 subjects. However, several experimental data are not high-quality. For instance, according to subject 1, $w(0.05) = 0.30$ and $w(0.10) = 0.28$. In other words, subject 1 deemed that the probability weight of 0.05 is bigger than the probability weight of 0.10 under prospect theory or cumulative prospect theory. According to the monotonicity of preference relation, the preference of subject 1 at $p = 0.10$ is not reasonable. Therefore, for subject 1, the preference points (0.10, 0.28), (0.50, 0.30) and (0.60, 0.45) and (0.90, 0.60) are abandoned to obtain high-quality EPPs. The abandoned EPPs are shown in bold in Table 5.
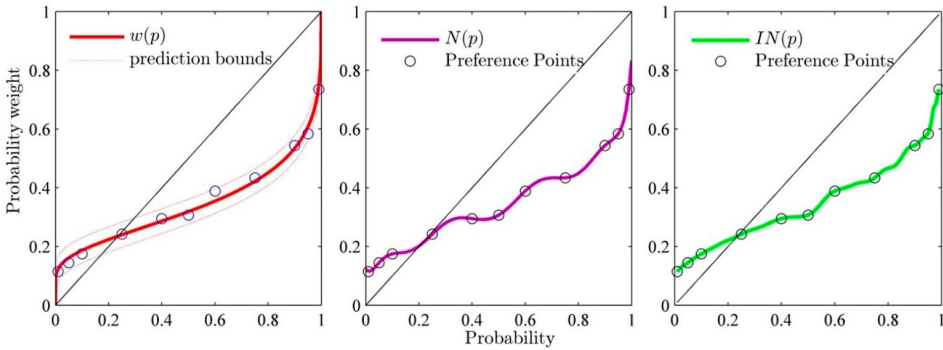

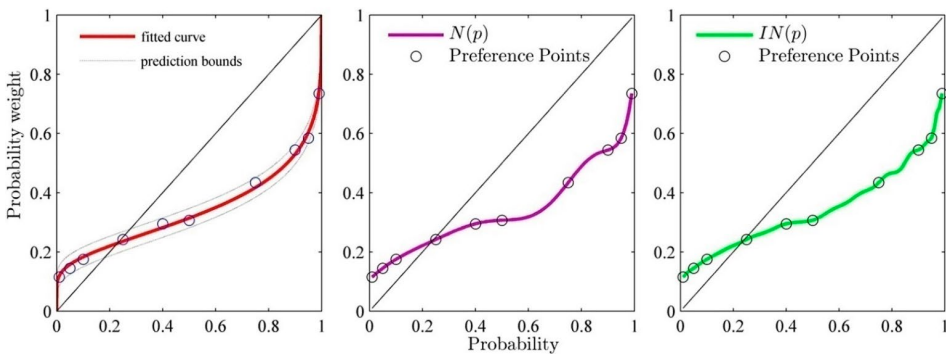Figure 10. PWF curves based on aggregate data

Figure 11. PWF curves based on aggregate data except for preference points (0.6, 0.389)

Table 5. Experimental data from Gonzalez and Wu (1999)

Subject	Probability p and probability weight $w(p)$										
	0.01	0.05	0.10	0.25	0.40	0.50	0.60	0.75	0.90	0.95	0.99
1	0.22	0.30	**0.28**	0.38	0.44	**0.30**	**0.45**	0.58	**0.60**	0.70	0.82
2	0.18	0.30	0.42	**0.54**	0.52	**0.50**	0.53	0.62	0.84	0.92	0.96
3	0.34	0.36	**0.52**	0.54	0.56	0.62	**0.56**	0.68	0.76	0.80	0.85
4	0.11	**0.39**	0.20	**0.31**	0.24	**0.31**	0.25	0.29	0.32	0.45	0.50
5	0.17	0.31	0.32	0.42	**0.39**	0.46	0.47	0.52	0.62	0.71	0.82
6	0.11	0.14	0.23	0.35	0.50	0.59	0.69	0.79	0.92	0.96	0.99
7	0.31	0.33	**0.39**	0.36	0.42	**0.35**	0.47	**0.46**	0.63	0.64	0.65
8	0.38	**0.23**	**0.22**	0.39	**0.22**	**0.23**	0.42	0.43	0.48	0.58	0.82
9	0.04	0.10	0.12	0.25	0.33	0.39	0.46	0.66	0.82	0.87	0.93
10	0.25	0.28	0.36	0.42	0.52	0.53	0.56	0.60	0.74	0.80	0.87

As shown in Figure 12, each individual participant exhibited a pattern according to the elicitation EPPs from Table 5 by using both two-parameter model and nonparametric numerical model. That is, the inverse S-shape of PWF, overweighting for small probability and underweighting for big probability appear to be a regularity that holds for most of individual subjects (except subjects 6 and 8). However, substantial heterogeneity is found in different individual subjects because of the differences of risk preferences.

First, using Prelec's two-parameter model $w(p)$ to approximate high-quality experimental data, the parameters correspond to the mathematic properties of curvature and elevation. Given that curvature and elevation appear to vary somewhat independently across participants, the model could fit several individual subjects such as subjects 1, 3, 4, 5, 6, 7, 9, and 10. At the same time, subjects 1, 5, 7 and 8 appear to cross the identity line at roughly the same level, yet they exhibit different degrees of curvature. Subjects 3 shows predominantly overweighting probability (relative to the identity line), whereas Subject 9 predominantly underweights probability; both participants exhibit roughly the same degree of curvature, but not elevation. Subject 6 always overweights probability, it does not satisfy the properties of inverse S-shape of PWF and underweighting for large probability, thus, subject 6 is always risk-seeking in situation.

Second, the nonparametric numerical approach is used to approximate PWF $IN(p)$ (Eq. (45)). For instance, subject 1 is linear in the range [0.05, 0.95], but subject 7 approaches linearity in the entire range, while others are nonlinear. Note that subjects 4 and 9 show predominantly underweighting probability (relative to the identity line); whereas, Subject 3 shows predominantly overweighting probability, and subject 6 shows absolutely overweighting probability; that finding proves that different participant exhibits different risk preference even if they are in the same environment. In the additive, subjects 2, 5 and 10 exhibit an inverse S-shaped PWF. Subject 8 is failure to be described both using parameter approach and using nonparametric numerical approach. Thus, the experimental data of subject 8 have low reliability.

Finally, the parameter and nonparametric numerical approach is compared at individual level, and the conclusions are similar to those at aggregate level are also obtained. For all
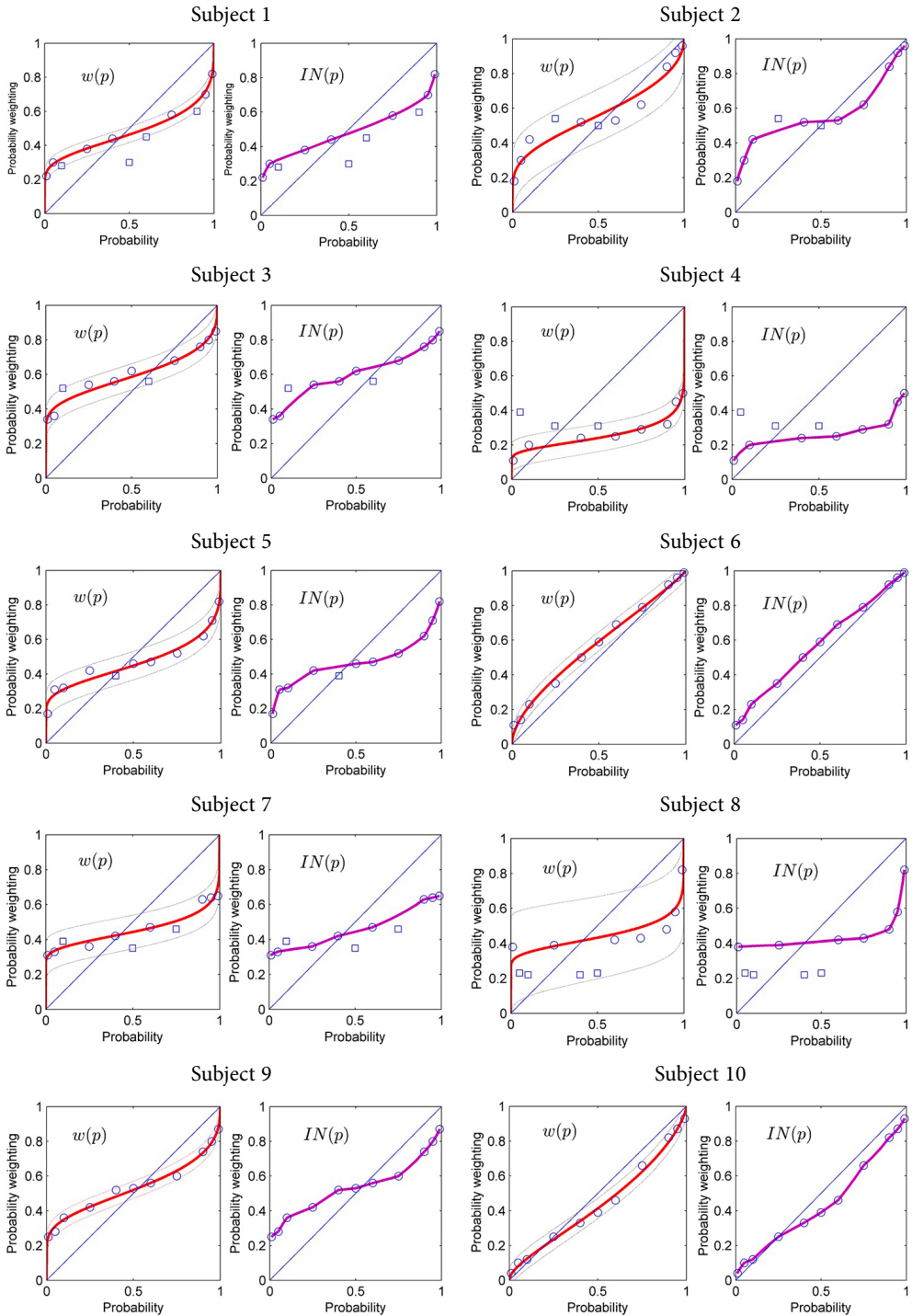
Figure 12. Fits of PWF for each individual participant using the parameter approach
and nonparametric numerical approach

of individual subjects, Subjects 1, 2, 4, 5 and 10 could yield a good PWF pattern both for parameter approach and for nonparametric numerical approach under the condition of high-quality EPPs. However, while the two-parameter PWF model fails to capture the pattern of subjects 2 and 8, yet the nonparametric numerical PWF model could do so successfully. Therefore, the proposed non- parameter numerical approach is a good approximation method to estimate the risk preferences of decision-maker in individual data.

On the whole, Empirical analysis provides that the proposed nonparametric numerical approach is a good method of building a nonparametric numerical PWF model that reflect and predict the preferences of decision-maker both in aggregate data and in individual data. The proposed model is required to go through all elicitation EPPs, which means that it could better reflect and predict the risk preferences of decision-maker than parameter models. However, the collected elicitation EPPs of the proposed approach has a higher quality requirement than parameter model.

More specifically, aggregate data are obtained by merging with the preferences of most decision-makers, these data are high-quality data in general. However, individual data are obtained by every decision-maker, they are not high-quality data because each individual decision-maker is faced with facing with the decision-making environment, economic conditions and cultural factors. Thus, individual data need to be processed by abandoning several non-high-quality EPPs such as the EPPs shown in bold in Table 5.

Conclusions

Different people have different risk attitudes, the same people who face different environments also have different risk attitudes, and different risk attitudes reflect different preferences. To estimate and measure these preferences, a nonparametric numerical approach is proposed in this paper. This nonparametric numerical approach is presented by combining parameter model and nonparametric eliciting approach. Empirical research at aggregate individual level has demonstrated that using the proposed approach to approximate PWF model could better reflect and predict the preferences of decision-maker without assuming a specific parametric form.

This paper not only offers a nonparametric numerical approach, but also provides some empirical researches both at aggregate level and at individual level. A nonparametric numerical approach allows researchers to infer the economic characteristics of an individual subject from the obtained preference information not only at aggregate subject level but also at individual subjects level. The non-linear PWF are shown based on empirical analysis. Moreover, the PWF approximated by the nonparametric numerical approach exhibits several characteristics such as an inverse S-shape and the "forgotten" fourfold pattern of risk preferences. In the actual enterprise management, the nonparametric PWF model proposed in this paper can effectively depict the risk preference of decision makers, and provide reliable theoretical and model basis for decision makers.

Of course, the use of the proposed nonparametric approach to approximate PWF has some disadvantages. For instance, when the collected experimental data are not high-quality, in reflecting and predicting the preferences of decision-maker is difficult for the built model, which has a significant effect given the collected experimental data. Moreover, compared with the parameter approach, the nonparametric numerical approach is more complex, and the

obtained curves are less observable and smooth. To solve these problem, non-high-quality experimental data were abandoned by determining whether these experimental data are away from the fitted two-parameter model (i.e., Prelec's PWF model) in this paper.

Future research

This paper focuses on proposing a nonparametric numerical approach to approximate PWFs for reflecting and predicting the risk preferences of decision-maker. To obtain the experimental data, aggregate and individual PWFs were elicited through the C-E method defined in Definition 4. However, some other elicitation methods, such as the tradeoff method, can be applied to elicit PWF to obtain several high-quality preferences information.

Although this paper focuses mainly on economic decision-making, the nonparametric numerical approach can be applied in other decision-making areas, such as, medical decision-making and behavior decision-making with applications to engineering management. Decision-making reflects the behavior of people, and people who have different genders, economic situations and come from different regions and countries have different risk preferences. The proposed model could be suitable for varied preferences with different high-quality preferences information.

Acknowledgements

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